LDE of 2nd & Higher Order

⊛ Linear DE

A linear DE of n^{th} order is that in which the dependent variable & its derivatives occur only in first degree & are not multiplied together.

Non-Homogeneous LDE of order n

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = Q \quad \text{--- (1)}$$

$P_0 \neq 0$

where P_i's are constants and Q is a fxn. of x or constt.

Homogeneous LDE of order n

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0$$

⊛ Differential operator D

$D \equiv \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, & so on

$\therefore$ (1) $\Rightarrow$

$$(P_0 D^n + P_1 D^{n-1} + \cdots + P_n) y = Q$$

$$f(D)\, y = Q
\quad \text{--- (2)}$$

Its complete soln. is

$$\boxed{y = C.F. + P.I.}$$

From solving $f(D)\, y = 0$

By solving $\frac{1}{f(D)} Q$

(*) **Auxillary Eqn. (A.E.)**

It is the eqn. obtained by equating to zero the symbolic coeff. of y

i.e. $f(D) = 0$.

(*) **Inverse Operator** $\frac{1}{f(D)}$.

$\frac{1}{f(D)} Q$ is that fnx. of x, free from arbitrary constts, which when operated upon $f(D)$ gives Q.

$\therefore \quad f(D) \left\{ \frac{1}{f(D)} Q \right\} = Q$

$\therefore \quad f(D)$ & $\frac{1}{f(D)}$ are inverse operators.

NOTE ① $\Rightarrow$ $\frac{1}{f(D)} Q$ is P.I. of $f(D)y = Q$

NOTE ② $\Rightarrow$ $\frac{1}{D} X = \int X \, dx$

NOTE ③ $\Rightarrow$ $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx$

$$\begin{array}{c|cccc} -1 & 9 & 3 & -5 & 1 \\ & & -9 & 6 & -1 \\ \hline & 9 & -6 & 1 & \lfloor 0 \end{array}$$

Ques. $9y''' + 3y'' - 5y' + y = 0$.

Soln. $(9D^3 + 3D^2 - 5D + 1)\,y = 0$

A.E. $9D^3 + 3D^2 - 5D + 1 = 0$

$\Rightarrow (D+1)(3D-1)^2 = 0$

$\Rightarrow D = -1, \frac{1}{3}, \frac{1}{3}$

$\therefore$ C.S. is $y = c_1 e^{-x} + (c_2 + c_3 x)\,e^{x/3}$.

Ans.

Ques. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 29y = 0$, given that

when $x = 0$, $y = 0$ & $\frac{dy}{dx} = 15$.

Soln. $(D^2 + 4D + 29)y = 0$

A.E. $D^2 + 4D + 29 = 0$

$\Rightarrow D = \frac{-4 \pm \sqrt{16 - 116}}{2}$

$D = -2 \pm 5i$

$\therefore$ C.S. is

$y = e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)$ —— (1)

$\frac{dy}{dx} = -2e^{-2x}(c_1 \cos 5x + c_2 \sin 5x) + e^{-2x}(-5c_1 \sin 5x + 5c_2 \cos 5x)$ —— (2)

When $x = 0$, $y = 0 \Rightarrow c_1 = 0$

When $x = 0$, $\frac{dy}{dx} = 15 \Rightarrow 15 = -2c_1 + 5c_2$

$\Rightarrow c_2 = 3$

$\therefore$ C.S. is $y = 3e^{-2x}\sin 5x$

Ans.

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Ques. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

Soln. $(D^4 + 8D^2 + 16)y = 0$

A.E. is $D^4 + 8D^2 + 16 = 0$

$(D^2+4)^2 = 0$

$\Rightarrow D = \pm 2i, \pm 2i$

$\therefore$ C.S. is

$y = (c_1 + c_2 x)\cos 2x + (c_1 + c_2 x)\sin 2x$

Ans.

Ques. $(D^2+1)^3(D^2+D+1)^2 y = 0.$

Soln. A.E. $(D^2+1)^3(D^2+D+1)^2 = 0$

$D = \pm i, \pm i, \pm i, \frac{-1\pm\sqrt{3}i}{2}, \frac{-1\pm\sqrt{3}i}{2}$

$\therefore$ C.S. is

$y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$
$+ e^{-x/2}\left[(c_7 + c_8 x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}x}{2}\right]$

Ans.

Ques. Solve $(D-2)^2 = 8(e^{2x} + \sin 2x + x^2)$.

Soln. A.E. $(D-2)^2 = 0$

$D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)e^{2x}$

P.I. $= \frac{1}{(D-2)^2}\left[8(e^{2x} + \sin 2x + x^2)\right]$ — (1)

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Now $\frac{1}{(D-2)^2} e^{2x} = x \frac{1}{2(D-2)} e^{2x}$ D=2 gives failure

$= x^2 \frac{1}{2} e^{2x}$ D=2 gives failure

Now $\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2-4D+4} \sin 2x$

$= \frac{1}{-4-4D+4} \sin 2x$ $D^2 = -2^2$

$= \frac{-1}{4} \frac{1}{D} \sin 2x$

$= \frac{1}{8} \cos 2x$

Now $\frac{1}{(D-2)^2} x^2 = \frac{1}{(2-D)^2} x^2$

$= \frac{1}{4\left(1-\frac{D}{2}\right)^2} x^2$

$= \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2} x^2$

$= \frac{1}{4}\left(1+D+\frac{3}{4}D^2+ \cdots\right) x^2$

$= \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$.

$\therefore$ (1) $\Rightarrow$ $P.I. = 8\left[\frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}\right]$.

$\therefore$ Complete solution is

$y = C.F. + P.I.$

$y = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$

Ans.

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Ques. $\frac{d^2y}{dx^2} + 4y = x \sin 2x.$

Soln. $(D^2+4)y = x \sin 2x$

A.E. $D^2+4=0 \Rightarrow D = \pm 2i$

$C.F = c_1 \cos 2x + c_2 \sin 2x.$

$P.I. = \frac{1}{D^2+4} x \sin 2x$

$= \text{Img. part of } \frac{1}{D^2+4} x e^{2ix}$

$= \text{Img. part } e^{2ix} \frac{1}{(D+2i)^2+4} x$

$= \text{Img. part } e^{2ix} \frac{1}{4Di + D^2} x$

$= \text{Img. part } e^{2ix} \frac{1}{4Di\left[1 - \frac{iD}{4}\right]} x$

$= \text{Img. part } e^{2ix} \frac{1}{4Di} \left[1 - \frac{iD}{4}\right]^{-1} x$

$= \text{Img. part } e^{2ix} \frac{1}{4Di} \left(1 + \frac{iD}{4}\right) x$

$= \text{Img. part } e^{2ix} \frac{1}{4Di} \left(x + \frac{i}{4}\right)$

$= \text{Img. part } \frac{-ie^{2ix}}{4} \left(\frac{x^2}{2} + \frac{ix}{4}\right)$

$P.I. = \frac{-x^2}{8} \cos 2x + \frac{x}{16} \sin 2x.$

$\therefore$ C.S. is $y = C.F. + P.I.$

$y = c_1 \cos 2x + c_2 \sin 2x - \frac{x^2}{8} \cos 2x + \frac{x}{16} \sin 2x.$

Ans.

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Ques. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^{x} \sin x$.

Soln. $D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$

$C.F. = (c_1 + c_2 x) e^{x}$.

$P.I. = \frac{1}{(D-1)^2} x e^{x} \sin x$

$= e^{x} \frac{1}{(D+1-1)^2} x \sin x$ $\qquad D \to D+1$

$= e^{x} \frac{1}{D^2} x \sin x$

$= e^{x} \frac{1}{D} \left[-x\cos x + \int \cos x \, dx \right]$

$= e^{x} \frac{1}{D} \left[-x\cos x + \sin x \right]$

$= e^{x} \left(-x \sin x + \int \sin x \, dx - \cos x \right)$

$= e^{x} \left(-x \sin x - 2\cos x \right)$

$P.I. = -e^{x} (x \sin x + 2\cos x)$.

$\therefore$ C.S. is

$y = C.F. + P.I.$

$y = (c_1 + c_2 x) e^{x} - e^{x} (x \sin x + 2\cos x)$

Ans.

Ques. $(D^2 + 2D + 2) y = e^{-x} \sec x$.

Soln. $D^2 + 2D + 2 = 0 \Rightarrow D = -1 \pm i$

$C.F. = e^{-x} (c_1 \cos x + c_2 \sin x)$.

$P.I. = \frac{1}{D^2 + 2D + 2} e^{-x} \sec x$

$= e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 2} \sec x$.

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$$P.I. = e^{-x} \frac{1}{D^2+1} \sec x$$

$$= e^{-x} \frac{1}{(D+i)(D-i)} \sec x .$$

$$\frac{1}{(D+i)(D-i)} = \frac{A}{D+i} + \frac{B}{D-i} \Rightarrow 1 = A(D-i) + B(D+i)$$

$$D = i \Rightarrow B = \frac{1}{2i}, \quad D = -i \Rightarrow A = -\frac{1}{2i}$$

$$P.I. = \frac{e^{-x}}{2i}\left[\frac{-1}{D+i} + \frac{1}{D-i}\right] \sec x$$

Now $\frac{1}{D+i} \sec x = e^{-ix} \int e^{ix} \sec x \, dx$

$$= e^{-ix} \int (\cos x + i \sin x) \sec x \, dx$$

$$= e^{-ix} \int (1 + i \tan x) \, dx$$

$$= e^{-ix} [x - i \log \cos x]$$

IIIly

$$\frac{1}{D-i} \sec x = e^{ix} [x + i \log \cos x] .$$

$$\therefore P.I. = \frac{e^{-x}}{2i}\left[-e^{-ix}(x - i \log \cos x) + e^{ix}(x + i \log \cos x)\right]$$

$$= \frac{e^{-x}}{2i}\left[x(e^{ix} - e^{-ix}) + i \log \cos x \, (e^{ix} - e^{-ix})\right]$$

$$= \frac{e^{-x}}{2i}\left(2xi \sin x + 2i \cos x \log \cos x\right)$$

$$= e^{-x}(x \sin x + \cos x \log \cos x) .$$

$\therefore$ C.S. is

$$y = C.F + P.I.$$

⊛ Method of Variation of Parameters to find P.I.

Let linear eqn. of 2^{nd} order with constt. coefficients be

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X \quad \text{—— (1)}$$

Let $y = c_1 y_1 + c_2 y_2$ be its C.F. so that y_1 & y_2 satisfy $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$ —— (2)

Let P.I. of (1) be $y = uy_1 + vy_2$ —— (3)

where u, v are unknown fns. of x.

Differentiating (3) w.r.t. x

$$y' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

Let $u'y_1 + v'y_2 = 0$ —— (4)

$\therefore$ $y' = uy_1' + vy_2'$ —— (5)

Differentiating (5) w.r.t. x

$$y'' = u'y_1' + uy_1'' + v'y_2' + vy_2'' \quad \text{—— (6)}$$

Put (3), (5) & (6) in (1)

$$u'y_1' + uy_1'' + v'y_2' + vy_2'' + a_1(uy_1' + vy_2') + a_2(uy_1 + vy_2) = X$$

$$\Rightarrow u(y_1'' + a_1y_1' + a_2y_1) + v(y_2'' + a_1y_2' + a_2y_2) + u'y_1' + v'y_2' = X$$

$\Rightarrow$ $u'y_1' + v'y_2' = X$ —— (7)

$\because$ y_1 & y_2 satisfy (2)

Solving (4) & (7)

$$u' = -\frac{y_2 X}{W}, \quad v' = \frac{y_1 X}{W}$$

where $\boxed{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$ is Wronskian of y_1, y_2

Integrating,

$$\boxed{u = -\int \frac{y_2 X}{W}\, dx}, \quad \boxed{v = \int \frac{y_1 X}{W}\, dx}$$

$\therefore$ (3) $\Rightarrow$ $\boxed{y = u y_1 + v y_2}$

P.I. is $y = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$.

$\longrightarrow$ As the solution is obtained by varying arbitrary constts. c_1, c_2 of the C.F., the method is known as variation of parameters.

Ques. Solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ by V.O.P

Soln. $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$.

A.E. $D^2 - 6D + 9 = 0 \Rightarrow D = 3, 3$

C.F. $= (c_1 + c_2 x)e^{3x}$.

Let $y_1 = e^{3x}$, $y_2 = x e^{3x}$, $X = \frac{e^{3x}}{x^2}$.

$$W = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & (1+3x)e^{3x} \end{vmatrix} = e^{6x}$$

P.I. $= u y_1 + v y_2$

where $u = -\int \frac{y_2 X}{W}\, dx$, $v = \int \frac{y_1 X}{W}\, dx$.

Now $u = -\int \frac{x e^{3x}}{e^{6x}} \frac{e^{3x}}{x^2}\, dx$

$= -\int \frac{1}{x}\, dx \Rightarrow u = -\log x$

$$v = \int \frac{y_1 X}{W} = \int \frac{e^{3x}}{e^{6x}} \frac{e^{3x}}{x^2} dx = \int \frac{dx}{x^2}$$

$$v = \frac{-1}{x}$$

$$\therefore \quad P.I. = -e^{3x} \log x - e^{3x}.$$
$$= -e^{3x}(1+\log x)$$

$\therefore$ Complete soln. is

$$y = C.F. + P.I.$$

$$y = (c_1 + c_2 x) e^{3x} - e^{3x}(1+\log x).$$

Ans.

Ques. Solve by using method of V.O.P

$$\frac{d^2y}{dx^2} + 4y = \tan 2x.$$

Soln. $(D^2+4)y = \tan 2x$

A.E. $D^2+4 = 0 \Rightarrow D = \pm 2i$

$$C.F. = c_1 \cos 2x + c_2 \sin 2x$$

Let $y_1 = \cos 2x$, $y_2 = \sin 2x$, $X = \tan 2x.$

$$P.I. = u y_1 + v y_2$$

where $u = -\int \frac{y_2 X}{W} dx$, $v = \int \frac{y_1 X}{W} dx$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Now $u = -\int \frac{\sin 2x \tan 2x}{2} dx$

$$= -\frac{1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx$$

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$$u = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$= \frac{-1}{2}\left[\frac{1}{2}\log(\sec 2x + \tan 2x) - \frac{1}{2}\sin 2x\right]$$

$$u = \frac{-1}{4}\log(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x.$$

Now $v = \int \frac{y_1 X}{W}\,dx$

$$= \int \frac{\cos 2x \tan 2x}{2}\,dx$$

$$= \frac{1}{2}\int \sin 2x\,dx = \frac{-\cos 2x}{4}.$$

$\therefore$ P. I. $= uy_1 + vy_2$

$$= -\frac{\cos 2x}{4}\log(\sec 2x + \tan 2x) +$$

$$\frac{1}{4}\sin 2x \cos 2x - \frac{1}{4}\sin 2x \cos 2x$$

$$\text{P. I.} = -\frac{\cos 2x}{4}\log(\sec 2x + \tan 2x).$$

$\therefore$ Complete soln. is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{\cos 2x}{4}\log(\sec 2x + \tan 2x)$$

Ans.

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Ques $(D^2+1)y = \operatorname{cosec} x \cot x$.

Soln: $D^2+1 = 0 \Rightarrow D = \pm i$

$C.F. = c_1 \cos x + c_2 \sin x$

Here $y_1 = \cos x$, $y_2 = \sin x$, $X = \operatorname{cosec} x \cot x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Let $P.I. = u y_1 + v y_2$

where $u = -\int \frac{y_2 X}{W}\,dx$, $v = \int \frac{y_1 X}{W}\,dx$.

$$\therefore u = -\int \sin x \cdot \operatorname{cosec} x \cot x \, dx$$

$$= -\int \cot x \, dx = -\log(\sin x).$$

$$v = \int \cos x \operatorname{cosec} x \cot x \, dx$$

$$= \int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\cot x - x$$

$P.I. = -\cos x \log(\sin x) - (\cot x + x)\sin x$.

$\therefore$ Complete soln. is

$$y = C.F. + P.I.$$

$$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sin x) - (\cot x + x)\sin x$$

Ans.

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(*) Solution of LDE with variable coefficients.

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = S$$

where P, Q, R, S are fns. of x.

$$(PD^2 + QD + R)y = S$$

Then factorise the LHS into two linear factors & hence operate these on y.

NOTE $\Longrightarrow$ $(D-1)(xD+1) \neq (xD+1)(D-1)$.

$\because (D-1)(xD+1) = D(xD+1) - xD - 1$

$= D + xD^2 + D - xD - 1$

$= xD^2 + (2-x)D - 1$

But $(xD+1)(D-1) = xD^2 + (1-x)D - 1$

Ques. Solve: $(x+1)y'' + (x-1)y' - 2y = 0$.

Soln. Symbolic form is

$$[(x+1)D^2 + (x-1)D - 2]y = 0 \quad \text{--- (1)}$$

$\Rightarrow (xD^2 + D^2 + xD - D - 2)y = 0$

$\Rightarrow [xD(D+1) + D^2 - D - 2]y = 0$

$\Rightarrow [xD(D+1) + (D+1)(D-2)]y = 0$

$\Rightarrow (xD + D - 2)(D+1)y = 0$

$\Rightarrow [(x+1)D - 2](D+1)y = 0 \quad \text{--- (2)}$

Let $(D+1)y = v \quad \text{--- (3)}$

$\int uv = u\int v - u'\iint v + u''\iiint v - \ldots$

(2) $\Rightarrow$ $[(x+1)D-2]v = 0$

$\Rightarrow (x+1)\dfrac{dv}{dx} - 2v = 0$

$\Rightarrow \dfrac{dv}{v} - \dfrac{2dx}{(x+1)} = 0$

$\Rightarrow v = c(x+1)^2$

(3) $\Rightarrow$ $(D+1)y = c(x+1)^2$

$\dfrac{dy}{dx} + y = c(x+1)^2$

which is linear in x

$I.f. = e^{\int dx} = e^x$

$\therefore$ Soln. is

$ye^x = c\int (x+1)^2 e^x dx + c'$

$= c\left[(x+1)^2 e^x - 2(x+1)e^x + 2e^x\right] + c'$

$\Rightarrow y = c(x^2+1) + c'e^{-x}$

Ans.

Ques. $[(x+3)D^2 - (2x+7)D + 2]y = (x+3)^2 e^x$. (1)

Soln.

$xD^2 + 3D^2 - 2xD - 7D + 2$

$xD(D-2) + 3D^2 - 7D + 2$

$xD(D-2) + (3D-1)(D-2)$

$(xD + 3D - 1)(D-2)$

$[(x+3)D - 1][D-2]y$

$$[(x+3)D-1][D-2]y = (x+3)^2 e^x \quad \text{---} (2)$$

Let $(D-2)y = v$ --- (3)

$$[(x+3)D-1]v = (x+3)^2 e^x$$

$$(x+3)\frac{dv}{dx} - v = (x+3)^2 e^x$$

$$\frac{dv}{dx} - \frac{v}{x+3} = (x+3)e^x$$

which is linear in x & v

$$I.F. = e^{-\int \frac{dx}{x+3}} = e^{-\log(x+3)} = \frac{1}{x+3}.$$

Soln. is

$$\frac{v}{x+3} = \int \frac{(x+3)e^x}{x+3}\,dx + c$$

$$v = (x+3)(e^x + c)$$

$\therefore$ (3) $\Rightarrow$ $\frac{dy}{dx} - 2y = (x+3)(e^x + c)$

which is linear in x & y

$$I.F. = e^{-\int 2dx} = e^{-2x}$$

Soln. is

$$y e^{-2x} = \int (x+3)(e^x + c)e^{-2x}\,dx + c'$$

$$= \int (xe^{-x} + cxe^{-2x} + 3e^{-x} + 3ce^{-2x})\,dx + c'$$

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$$ye^{-2x} = -xe^{-x} - e^{-x} + c\left[\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right] + \frac{3e^{-x}}{-1} + 3c\frac{e^{-2x}}{-2} + c'$$

$$= -xe^{-x} - e^{-x} - \frac{cxe^{-2x}}{2} - \frac{ce^{-2x}}{4} - 3e^{-x} - \frac{3}{2}ce^{-2x} + c'$$

$$ye^{-2x} = \cancel{x(e} - e^{-x}(x+1) - \frac{3cxe^{-2x}}{2} - \frac{7}{4}ce^{-2x} - 3e^{-x} + c'$$

$$y = -(x+1)e^{x} - \frac{c}{2}x - \frac{7}{4}c - 3e^{x} + c'e^{2x}$$

$$y = c'e^{2x} - (x+4)e^{x} + (2x+7)c_1 \quad , \quad c_1 = \frac{-c}{4}$$

Ans.

Cauchy's Homogeneous Linear Eqn.

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = Q$$

where P_i's are constt. & Q is a fnx. of x.

S.F.

$$(P_0 x^n D^n + P_1 x^{n-1} D^{n-1} + \cdots + P_n)y = Q$$

Put $x = e^z \Rightarrow z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow xDy \equiv \theta y$$

where $\theta = \frac{d}{dz}$

$$\boxed{xD \equiv \theta}$$

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$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \left(\frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\right)\cdot\frac{1}{x}$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2}$$

$$x^2D^2y = -\theta y + \theta^2 y$$

$$\Rightarrow \boxed{x^2D^2 \equiv \theta(\theta-1)}$$

Illy $x^3D^3 = \theta(\theta-1)(\theta-2)$ & so on.

Ques. $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$.

Soln. $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x+\frac{1}{x}\right)$ —①

Put $x = e^z \Rightarrow z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$xD \equiv \theta$, $x^2D^2 \equiv \theta(\theta-1)$

$x^3D^3 \equiv \theta(\theta-1)(\theta-2)$

① $\Rightarrow$

$$[\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2]y = 10(e^z + e^{-z})$$

$\theta^2 - 3\theta + 2$

$$[\theta^3 - 3\theta^2 + 2\theta + 2\theta^2 - 2\theta + 2]y = 10(e^z + e^{-z})$$

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$$(\theta^3-\theta^2+2)y = 10(e^{z}+e^{-z}) \quad \text{—— (2)}$$

A.E. $\theta^3-\theta^2+2 = 0$

$(\theta+1)(\theta^2-2\theta+2) = 0$

$\theta = -1, \; 1 \pm i$

$\therefore$ C.F. $= c_1 e^{-z} + e^{z}(c_2\cos z + c_3 \sin z)$.

P.I. $= \dfrac{1}{\theta^3-\theta^2+2}\, 10(e^{z}+e^{-z})$

$= 10\left[\dfrac{1}{\theta^3-\theta^2+2}e^{z} + \dfrac{1}{\theta^3-\theta^2+2}e^{-z}\right]$

$= 10\left[\dfrac{1}{2}e^{z} + z\,\dfrac{1}{3\theta^2-2\theta}e^{-z}\right]$

$= 10\left(\dfrac{1}{2}e^{z} + \dfrac{z e^{-z}}{5}\right)$

$= 5e^{z} + 2ze^{z}$.

$\therefore$ Complete soln. is

$y = c_1 e^{-z} + e^{z}(c_2 \cos z + c_3 \sin z) + 5e^{z} + 2ze^{-z}$

$y = \dfrac{c_1}{x} + x\left(c_2\cos(\log x) + c_3 \sin(\log x)\right) + 5x + \dfrac{2\,\cancel{x}\log x}{x}$

Ans.

Ques. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$.

Soln. $[x^2D^2 - xD + 2]y = x\log x \quad \text{—— (1)}$

Put $x = e^{z} \Rightarrow z = \log x \Rightarrow \dfrac{dz}{dx} = \dfrac{1}{x}$.

$\therefore\; xD \equiv \theta, \quad x^2D^2 \equiv \theta(\theta-1)$

① $\Rightarrow$ $[\theta(\theta-1) - \theta + 2]y = ze^z$

$(\theta^2 - 2\theta + 2)\,y = z\,e^z$

A.E. $\quad \theta^2 - 2\theta + 2 = 0$

$\Rightarrow \quad \theta = 1 \pm i$

$\therefore$ C.F. $= (c_1 \cos z + c_2 \sin z)\, e^z$

P.I. $= \dfrac{1}{\theta^2 - 2\theta + 2}\, z e^z$

$= e^z \dfrac{1}{(\theta+1)^2 - 2(\theta+1) + 2}\, z$

$= e^z \dfrac{1}{1+\theta^2}\, z$

$= e^z (1+\theta^2)^{-1}\, z$

$= e^z (1 - \theta^2 + \theta^4 - \dots)\, z$

$= e^z (z)$

$\therefore$ Complete soln. is

$y = (c_1 \cos z + c_2 \sin z)e^z + ze^z$

$y = x\left(c_1 \cos(\log x) + c_2 \sin(\log x)\right) + x \log x.$

Ans.

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Ques: $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$.

Soln: $x^3 y''' + 2x^2 y'' - x y' + y = \frac{1}{x}$

$$(x^3 D^3 + 2x^2 D^2 - x D + 1) y = \frac{1}{x} \quad \text{—(1)}$$

Put $x = e^z \Rightarrow z = \log x$

$xD \equiv \theta$, $x^2 D^2 \equiv \theta(\theta-1)$, $x^3 D^3 = \theta(\theta-1)(\theta-2)$

(1) $\Rightarrow$

$$[\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) - \theta + 1] y = e^{-z}$$

$$(\theta^3 - 3\theta^2 + \cancel{2\theta} + 2\theta^2 - \cancel{2\theta} - \theta + 1) y = e^{-z}$$

$$(\theta^3 - \theta^2 - \theta + 1) y = e^{-z}$$

A.E. $\theta^3 - \theta^2 - \theta + 1 = 0$

$(\theta - 1)(\theta^2 - 1) = 0$

$\Rightarrow \theta = 1, 1, -1$

C.F. $= (c_1 + c_2 z) e^z + c_3 e^{-z}$.

P.I. $= \frac{1}{\theta^3 - \theta^2 - \theta + 1} e^{-z}$

$= z \frac{1}{3\theta^2 - 2\theta - 1} e^{-z} = \frac{z e^{-z}}{4}$.

$\therefore$ C.S. is $y =$ C.F. + P.I.

$$y = (c_1 + c_2 z) e^z + c_3 e^{-z} + \frac{z e^{-z}}{4}$$

$$y = x(c_1 + c_2 \log x) + \frac{c_3}{x} + \frac{\log x}{4x}.$$

Ans.

Ques: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$.

Soln: $(x^2D^2 + xD + 1)y = \log x \sin(\log x)$ —— ①

Put $x = e^z \Rightarrow z = \log x$

$xD \equiv \theta$, $x^2D^2 \equiv \theta(\theta-1)$

① ⇒

$[\theta^2 - \theta + \theta + 1]\, y = z \sin z$

$(\theta^2+1)\, y = z\sin z$ —— ②

A.E.: $\theta^2 + 1 = 0 \rightarrow \theta = \pm i$

C.F. $= c_1 \cos z + c_2 \sin z$

P.I. $= \frac{1}{\theta^2+1} z \sin z$

$=$ Img. part of $\frac{1}{\theta^2+1} z e^{iz}$

$=$ Imag. part of $e^{iz} \frac{1}{(\theta+i)^2+1} z$

$=$ Imag. part of $e^{iz} \frac{1}{2i\theta+\theta^2} z$

$=$ Img. part of $e^{iz} \frac{1}{2i\theta}\left(1 - \frac{i\theta}{2}\right)^{-1} z$

$=$ Img. part of $\frac{e^{iz}}{2i} \frac{1}{\theta}\left(1+\frac{i\theta}{2}\right) z$

$=$ Img. part of $\frac{-ie^{iz}}{2} \frac{1}{\theta}\left(z + \frac{i}{2}\right)$

$=$ Img. part of $\frac{-i}{2} e^{iz}\left(\frac{z^2}{2} + \frac{iz}{2}\right)$

$= \frac{-z^2}{4}\cos z + \frac{z \sin z}{4}$

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$\therefore$ Complete soln. is

$$y = C.F. + P.I.$$

$$y = c_1 \cos z + c_2 \sin z - \frac{z^2}{4}\cos z + \frac{z}{4}\sin z.$$

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x).$$

Ans.

Legendre's Linear Eqn:

$$P_0(a+bx)^n \frac{d^n y}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n y = Q$$

where P_i's are constts. & Q is a fnx. of x.

Put $a+bx = e^z \Rightarrow z = \log(a+bx)$

s.t. $(a+bx)D \equiv b\theta$

$(a+bx)^2 D^2 \equiv b^2\theta(\theta-1)$

$(a+bx)^3 D^3 \equiv b^3\theta(\theta-1)(\theta-2)$

and so on.

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Ques: $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x).$

Soln: $[(1+x)^2 D^2 + (1+x)D + 1]y = 4\cos\log(1+x)$ — ①

Put $1+x = e^z \Rightarrow z = \log(1+x)$.

$(1+x)D \equiv \theta$

$(1+x)^2 D^2 \equiv \theta(\theta-1)$.

① $\Rightarrow$

$[\theta^2 - \theta + \theta + 1]y = 4\cos z$

$(\theta^2+1)y = 4\cos z$

A.E. $\theta^2 + 1 = 0 \Rightarrow \theta = \pm i$

C.F. $= c_1\cos z + c_2 \sin z$.

P.I. $= \frac{1}{\theta^2+1} 4\cos z$

$= 4z \frac{1}{2\theta}\cos z$

$= 2z\sin z$.

$\therefore$ Complete soln. is

$y = c_1\cos z + c_2 \sin z + 2z\sin z$.

$y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] + 2\log(1+x)\sin\log(1+x)$

Ans.

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Ques. $(1+2x)^2 y'' - 6(1+2x) y' + 16y = 8(1+2x)^2$.

Soln. $\left[(1+2x)^2 D^2 - 6(1+2x) D + 16\right] y = 8(1+2x)^2$ ——— (1)

Put $1+2x = e^z \Rightarrow z = \log(1+2x)$

$(1+2x) D \equiv 2\theta$

$(1+2x)^2 D^2 = 4\theta(\theta-1)$

(1) $\Rightarrow$

$(4\theta^2 - 4\theta - 12\theta + 16) y = 18 e^{2z}$

$(4\theta^2 - 16\theta + 16) y = 18 e^{2z}$ ——— (2)

A.E. $\quad 4\theta^2 - 16\theta + 16 = 0$

$(4\theta - 8)(\theta - 2) = 0$

$\theta = 2, 2$

$C.F. = (c_1 + c_2 z) e^{2z}$

$P.I = \dfrac{1}{4\theta^2 - 16\theta + 16} 18 e^{2z}$

$= 18 \dfrac{1}{4\theta^2 - 16\theta + 16} e^{2z}$

$= 18 z \dfrac{1}{8\theta - 16} e^{2z}$

$= 18 z^2 \dfrac{1}{8} e^{2z} = \dfrac{9}{4} z^2 e^{2z}$

$\therefore$ C.S. is $\quad y = CF + PI$

$y = (c_1 + c_2 z) e^{2z} + \dfrac{9}{4} z^2 e^{2z}$

$y = [c_1 + c_2 \log(1+2x)](1+2x)^2 + \dfrac{9}{4}(1+2x)^2 [\log(1+2x$

Ans.

Simultaneous Linear Eqn. with constt. coeff.

Ques: Solve $\frac{dx}{dt} + 2y = e^{t}$ — (1) & $\frac{dy}{dt} - 2x = e^{-t}$. — (2)

Soln: Differentiating (2) w.r.t. t

$$\frac{d^2y}{dt^2} - 2\frac{dx}{dt} = -e^{-t}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2}\left(\frac{d^2y}{dt^2} + e^{-t}\right)$$

$$(1) \Rightarrow \frac{1}{2}\frac{d^2y}{dt^2} + \frac{1}{2}e^{-t} + 2y = e^{t}$$

$$\frac{d^2y}{dt^2} + 4y = 2e^{t} - e^{-t}$$

$$(D^2+4)y = 2e^{t} - e^{-t}$$

A.E. $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

C.F. $= c_1 \cos 2t + c_2 \sin 2t$

P.I. $= \frac{1}{D^2+4}\left(2e^{t} - e^{-t}\right)$

$= \frac{2}{5}e^{t} - \frac{1}{5}e^{-t}$

$$\therefore y = c_1\cos 2t + c_2 \sin 2t + \frac{2}{5}e^{t} - \frac{1}{5}e^{-t}.$$

$$\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{2}{5}e^{t} + \frac{1}{5}e^{-t} \quad (3)$$

Put (3) in (2)

$$-2c_1 \sin 2t + 2c_2\cos 2t + \frac{2}{5}e^{t} + \frac{1}{5}e^{-t} - e^{-t} = 2x$$

$$\Rightarrow x = -c_1 \sin 2t + c_2 \cos 2t + \frac{1}{5}e^{t} - \frac{2}{5}e^{-t}.$$

Ans.

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Ques. $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$

Given that $x = y = 0$ when $t = 0$.

Soln. $(D+5)x - 2y = t$ —— (1)

$2x + (D+1)y = 0$ —— (2)

$2(1) - (D+5)(2)$

$(D^2+6D+9)y = -2t$

A.E. $D^2+6D+9 = 0$

$(D+3)^2 = 0 \Rightarrow D = -3, -3$

$C.F. = (c_1 + c_2 t)e^{-3t}$

$P.I. = \frac{1}{9+6D+D^2}(-2t)$

$= \frac{-2}{9}\left[1+\left(\frac{2}{3}D+\frac{D^2}{9}\right)\right]^{-1} t$

$= \frac{-2}{9}\left[1-\left(\frac{2}{3}D+\frac{D^2}{9}\right)\right] t$

$= \frac{-2}{9}\left(t - \frac{2}{3}\right)$

$= \frac{-2}{9}t + \frac{4}{27}$

$y = (c_1 + c_2 t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}$ —— (3)

$\frac{dy}{dt} = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t} - \frac{2}{9}$ —— (4)

Put (3) + (4) in (2)

$2x + -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t} - \frac{2}{9} + c_1 e^{-3t} + c_2 t e^{-3t} - \frac{2}{9}t + \frac{4}{27} = 0$

$x = \left(c_1 - \frac{c_2}{2} + c_2 t\right)e^{-3t} + \frac{1}{9}t + \frac{1}{27}$ —— (5)

When $t = 0$, $x = 0$

$$⑤ \Rightarrow \quad c_1 - \frac{c_2}{2} + \frac{1}{27} = 0 \quad \text{—— } ⑥$$

When $t = 0$, $y = 0$

$$③ \Rightarrow \quad c_1 + \frac{4}{27} = 0 \Rightarrow c_1 = -\frac{4}{27}$$

$$⑥ \Rightarrow \quad c_2 = -\frac{2}{9}$$

$\therefore$ ⑤ & ③ become

$$x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(1+3t).$$

$$y = -\frac{2}{27}(2+3t)e^{-3t} + \frac{2}{27}(2-3t)$$

Ans.

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