ODE of FIRST ORDER

⊛ ODE

which involve only one independent variable and the differential coefficients w.r.t. it.

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \quad \text{--- (1)}$$

$$y = x\frac{dy}{dx} + \frac{c}{dy/dx} \quad \text{--- (2)}$$

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\cdot\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \quad \text{--- (3)}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k.\frac{d^2y}{dx^2} \quad \text{--- (4)}$$

⊛ Order

It is the order of the highest order derivative occurring in the diff. eqn.

e.g. Eqns (1) + (4) are of 2nd order
(2) is of 1st order
(3) is of 3rd order.

⊛ Degree

It is the degree of the highest order derivative which occurs in the diff. eqn. provided the eqn. has been made free of the radicals + fractions as far as the derivatives are concerned.

e.g. (1) + (3) are of degree one.

(2) $\Rightarrow \quad y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c \quad \Rightarrow$ Degree = 2

(4) $\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2 \Rightarrow$ Degree $= 2$

(*) Solution of a D.E.

It is a relation, free from derivatives, b/w the variables which satisfies the eqn.

e.g. $y = c_1 \cos x + c_2 \sin x$ is soln. of the differential eqn. $\frac{d^2y}{dx^2} + y = 0$. —— (1)

(*) General (Complete) solution

No. of arbitrary constants $=$ Order of DE

e.g. $y = c_1 \cos x + c_2 \sin x$ is general soln. of $\frac{d^2y}{dx^2} + y = 0$ of 2nd order.

(*) Particular Solution

Obtained from its general solution by giving particular values to constants.

e.g. (1) $\Rightarrow$ $\left.\begin{array}{l} y = \cos x \\ y = \cos x - \sin x \end{array}\right]$ is particular soln. of $\frac{d^2y}{dx^2} + y = 0$.

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Ques. Obtain the differential eqns :-

(i) $y = A + Bx + Cx^2$

$$\frac{dy}{dx} = B + 2Cx \quad , \quad \frac{d^2y}{dx^2} = 2C$$

$$\boxed{\frac{d^3y}{dx^3} = 0} \Rightarrow \text{Ans.}$$

(ii) $y = A\cos 2t + B\sin 2t$

$$\frac{dy}{dt} = -2A\sin 2t + 2B\cos 2t$$

$$\frac{d^2y}{dt^2} = -4A\cos 2t - 4B\sin 2t$$

$$= -4(A\cos 2t + B\sin 2t)$$

$$= -4y$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} + 4y = 0} \quad \text{Ans.}$$

Ques. All circles of radius r whose centre lie on the x-axis.

Soln. $(x-a)^2 + y^2 = r^2$ —— (1)

$$2(x-a) + 2y\frac{dy}{dx} = 0 \quad \text{—— (2)}$$

$$x - a = -y\frac{dy}{dx} \quad \text{—— (3)}$$

(1) $\Rightarrow$ $y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2$

$$y^2 = \frac{r^2}{1+\left(\frac{dy}{dx}\right)^2} \quad \text{—— (4)}$$

$$① \Rightarrow \quad y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2$$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{r^2}{y^2}$$

$$\Rightarrow y^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = r^2$$

Ans

★ Solution of DE of 1st order & 1st degree.

① Variable seperable method

② $\frac{dy}{dx} = f(ax+by+c) \Rightarrow ax+by+c = t$

$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

③ Homogeneous eqns. $\Rightarrow y = vx$

④ Linear eqns.

⑤ Exact eqns.

Ques. $\frac{dy}{dx} = e^{2x+3y}$

Soln. $\frac{dy}{dx} = e^{2x} \cdot e^{3y}$

$$\frac{dy}{e^{3y}} = e^{2x}\,dx \Rightarrow e^{-3y}\,dy = e^{2x}\,dx$$

$$\Rightarrow \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c$$

$$\Rightarrow 3e^{2x} + 2e^{-3y} = c$$

Ans

Ques. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Soln. $\sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$

$$\frac{\sec^2 x \, dx}{\tan x} = -\frac{\sec^2 y}{\tan y} dy$$

Integrating

$$\log \tan x = -\log \tan y + \log c$$

$$\Rightarrow \tan x \cdot \tan y = c$$

Ans.

Ques. $$\frac{dy}{dx} = (4x + y + 1)^2$$

Soln. $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4$$

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\frac{dt}{t^2 + 4} = dx$$

$$\frac{1}{2} \tan^{-1} \frac{t}{2} = x + c'$$

$$\tan^{-1} \frac{(4x + y + 1)}{2} = 2x + c$$

$$\Rightarrow 4x + y + 1 = 2 \tan (2x + c)$$

Ans.

Ques: $(x+y)\,dx + (y-x)\,dy = 0$.

Soln: $$\frac{dy}{dx} = \frac{x+y}{x-y} \quad \text{———} \quad (1)$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$(1) \Rightarrow \quad v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$= \frac{1+v-v+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}$$

$$\frac{dv}{1+v^2} - \frac{1}{2}\frac{2v}{1+v^2}\,dv = \frac{dx}{x}$$

Integrating, we get

$$\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + c$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(1+\frac{y^2}{x^2}\right) = \log x + c$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\log(x^2+y^2) + \log x = \log x + c$$

$$\tan^{-1}\frac{y}{x} = \frac{1}{2}\log(x^2+y^2) + c$$

Ans.

(*) Exact Differential Eqn.

A D.E. of the form $Mdx + Ndy = 0$ is s.t.b exact if it can be obtained directly by differentiating the eqn. $u(x,y) = c$ which is its primitive i.e. if

$$du = Mdx + Ndy$$

where M and N are fns. of x & y.

Thm:- The necessary and sufficient condition for the differential eqn. $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Soln. Necessay Condition

The eqn. $Mdx + Ndy = 0$ will be exact if $du = Mdx + Ndy$.

But $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$$\therefore \frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Since $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

which is the reqd. necessay condition.

Sufficient Condition

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

T.P. $M dx + N dy = 0$ is exact.

Let $u = \int_{y\ \text{constt}} M dx$

$\Rightarrow M = \frac{\partial u}{\partial x}$ & $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}$

Since $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$

Also $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Integrating w.r.t. x keeping y constt.

$N = \frac{\partial u}{\partial y} + f(y)$

$\therefore M dx + N dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + f(y) dy$

$= du + d\left[\int f(y)\, dy\right]$

$= d\left[u + \int f(y)\, dy\right]$

which is an exact diff. eqn.

$\therefore M dx + N dy = 0$ is exact.

Hence proved.

NOTE $\Rightarrow$ Soln. of $M\,dx + N\,dy = 0$.

Since $M\,dx + N\,dy = d\left[u + \int f(y)\,dy\right]$

$\therefore$ $u + \int f(y)\,dy = c$ is soln.

$$\boxed{\Rightarrow \int_{y\ \text{constt.}} M\,dx + \int (\text{Terms of } N \text{ free from } x)\,dy = c}$$

Ques. Solve:

$$(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0 \quad \text{---} (1)$$

Soln.
$$M = y^2 e^{xy^2} + 4x^3$$
$$N = 2xy\,e^{xy^2} - 3y^2.$$

$$\frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2 e^{xy^2} \cdot 2xy$$
$$= 2y\,e^{xy^2} + 2xy^3\,e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2y\,e^{xy^2} + 2xy^3\,e^{xy^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow (1) \text{ is exact.}$$

$\therefore$ Soln. is

$$\int_{y\ \text{constt.}} (y^2 e^{xy^2} + 4x^3)\,dx - \int 3y^2\,dy = c$$

$$y^2 \frac{e^{xy^2}}{y^2} + x^4 - y^3 = c$$

$$\Rightarrow \quad e^{xy^2} + x^4 - y^3 = c$$

Ans.

Ques. $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ —— (1)

Soln. $(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{(1) is exact.}$$

Soln. is

$$\int (y\cos x + \sin y + y)\,dx + \int 0 \cdot dy = c$$

y constt.

$$y\sin x + x\sin y + xy = c$$

Ans.

Ques. $x\,dy + y\,dx + \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0$ —— (1)

Soln. $M = y - \dfrac{y}{x^2+y^2}$

$$N = x + \frac{x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = 1 - \frac{x^2+y^2 - y(2y)}{(x^2+y^2)^2} = 1 - \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = 1 + \frac{x^2+y^2 - x2x}{(x^2+y^2)^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$= 1 - \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{(1) is exact.}$$

$\therefore$ Soln. is

$$\int \left(y - \frac{y}{x^2+y^2} \right) dx + \int 0 \cdot dy = c$$

y constt.

$$xy - y \cdot \frac{1}{y} \tan^{-1} \frac{y}{x} = c$$

$$xy - \tan^{-1} \frac{y}{x} = c$$

Ans.

Ques. For what value of k, the differential eqn. $\left(1 + e^{kx/y}\right) dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

is exact?

Soln. $\frac{\partial M}{\partial y} = e^{kx/y} \left(\frac{-kx}{y^2} \right)$

$$\frac{\partial N}{\partial x} = \frac{e^{x/y}}{y}\left(1 - \frac{x}{y}\right) + e^{x/y}\left(\frac{-1}{y}\right)$$

Since given eqn. is exact, thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$e^{kx/y}\left(\frac{-kx}{y^2}\right) = \frac{e^{x/y}}{y} - \frac{x}{y^2} e^{x/y} - \frac{e^{x/y}}{y}$$

$$\Rightarrow k e^{kx/y} = e^{x/y}$$

which holds when $\boxed{k = 1}$

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Equations Reducible to Exact Eqns.

D.E. which are not exact can sometimes be made exact after multiplying by a factor (fnx. of x or y or both) called an <u>integrating factor</u>.

I.F. found by inspection

1. $y\,dx + x\,dy = d(xy)$
2. $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
3. $\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$.
4. $\dfrac{x\,dy - y\,dx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$.
5. $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
6. $\dfrac{x\,dy + y\,dx}{xy} = d\left[\log(xy)\right]$.
7. $\dfrac{x\,dy - y\,dx}{x^2-y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$
8. $\dfrac{x\,dy + y\,dx}{x^2+y^2} = d\left[\dfrac{1}{2}\log(x^2+y^2)\right]$.

Ques. Find the I.F. of $(y-1)\,dx - x\,dy = 0$ and hence solve it. —— (1)

Soln. $M = y-1$, $N = -x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{equ. is not exact.}$$

$$(1) \Rightarrow \quad y\,dx - dx - x\,dy = 0$$

$$\frac{y\,dx - x\,dy}{x^2} = \frac{dx}{x^2}$$

$$-\,d\left(\frac{y}{x}\right) = -\frac{1}{x} + c$$

$$y = 1 - cx$$

Ans.

Ques. $y(2xy + e^x)\,dx - e^x\,dy = 0$ —— (1)

Soln. $\dfrac{\partial M}{\partial y} = 4xy + e^x$

$\dfrac{\partial N}{\partial x} = -e^x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow (1) \text{ is not exact.}$$

$$(1) \Rightarrow \frac{y(2xy + e^x)\,dx}{y^2} - \frac{e^x}{y^2}\,dy = 0$$

$$\left(2x + \frac{e^x}{y}\right)dx - \frac{e^x}{y^2}\,dy = 0$$

Now this is exact equ.

$$\int_{y\ \text{const}} \left(2x + \frac{e^x}{y}\right) dx + \int 0\, dy = c$$

$$x^2 + \frac{e^x}{y} = c$$

Ans.

I.F. for Homogeneous eqn.

If $M\,dx + N\,dy = 0$ is a homogeneous eqn. in x & y, then

$$\text{I.F.} = \frac{1}{Mx + Ny} \quad ; \text{ provided } Mx + Ny \neq 0.$$

⟹ **Case of failure**

If $Mx + Ny = 0 \Rightarrow \frac{M}{N} = -\frac{y}{x}$

$$M\,dx + N\,dy = 0$$

$$\Rightarrow \frac{M}{N} dx + dy = 0 \Rightarrow -\frac{y}{x} dx + dy = 0$$

$$\Rightarrow -\frac{dx}{x} + \frac{dy}{y} = 0$$

$$\Rightarrow y = cx \text{ is reqd. soln.}$$

I.F. for an eqn. of the form $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$

& $\text{I.F.} = \frac{1}{Mx - Ny}$; provided $Mx - Ny \neq 0$.

⟹ **Case of failure**

$Mx - Ny = 0 \Rightarrow M/N = y/x$

$$M\,dx + N\,dy = 0 \Rightarrow \frac{y}{x}\,dx + dy = 0$$

$$\Rightarrow x\,dy + y\,dx = 0$$

$$\Rightarrow xy = c \text{ is reqd. soln.}$$

(*) For the eqn. $M\,dx + N\,dy = 0$

(i) If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$, then $I.F. = e^{\int f(x)\,dx}$

(ii) If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$, then $I.F. = e^{\int g(y)\,dy}$.

(*) If eqn. is of the form

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

$$I.F. = x^h y^k$$

where $\dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}$

$$\frac{c+h+1}{p} = \frac{d+k+1}{q}$$

Ques. $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$ —①

Soln. $M = 3xy^2 - y^3$, $N = xy^2 - 2x^2y$

$\frac{\partial M}{\partial y} = 6xy - 3y^2$, $\frac{\partial N}{\partial x} = y^2 - 4xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact

$$I.F. = \frac{1}{Mx + Ny}$$

$$= \frac{1}{3x^2y^2 - xy^3 + xy^3 - 2x^2y^2} = \frac{1}{x^2y^2}$$

$$① \Rightarrow \left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} - \frac{2}{y}\right)dy = 0$$

which is exact.

$\therefore$ Soln. is

$$\int_{y\ \text{constt.}} \left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \int \frac{-2}{y}\,dy = c$$

$$3\log x + \frac{y}{x} - 2\log y = c.$$

Ans.

Ques. $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$ —①

Soln. $M = x^2y^3 + xy^2 + y$

$N = x^3y^2 - x^2y + x$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1$

$\frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact.

$$I.F. = \frac{1}{Mx - Ny}$$

$$= \frac{1}{x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy}$$

$$= \frac{1}{2x^2y^2}$$

$\therefore$ ① $\Rightarrow$

$$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \left(\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right)dy = 0$$

which is exact.

$\therefore$ Soln. is

$$\int_{y\ \text{constt.}} \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx + \int \frac{-1}{2y}\,dy = C$$

$$\frac{xy}{2} + \frac{1}{2}\log x - \frac{1}{2xy} - \frac{1}{2}\log y = C$$

$$\Rightarrow xy + \log\frac{x}{y} - \frac{1}{xy} = C$$

Ans.

Ques. $(x^2+y^2+2x)\,dx + 2y\,dy = 0$. —— ①

Soln. $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ ① is not exact.

Now $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y}{2y} = 1 = f(x)$

$$I.F. = e^{\int f(x)\,dx} = e^{\int dx} = e^x.$$

$\therefore$ ① $\Rightarrow$

$$e^x(x^2+y^2+2x)\,dx + 2y e^x\,dy = 0$$

which is exact.

$\therefore$ Soln. is

$$\int_{y\text{ const}} \left[e^x(x^2+2x) + y^2 e^x\right] dx = C$$

$$(x^2+2x)e^x - \int (2x+2)e^x\,dx + y^2 e^x = C$$

$$(x^2+2x)e^x - (2x+2)e^x + 2e^x + y^2 e^x = C$$

$$x^2 e^x + \cancel{2xe^x} - \cancel{2xe^x} - \cancel{2e^x} + \cancel{2e^x} + y^2 e^x = C$$

$$e^x(x^2+y^2) = C$$

Ans.

Ques. $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$ — ①

Soln. $\dfrac{\partial M}{\partial y} = 4y^3+2$

$\dfrac{\partial N}{\partial x} = y^3-4$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ ① is not exact.

Now, $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3(y^3+2)}{y(y^3+2)} = \frac{-3}{y} = g(y)$$

$$I.f. = e^{\int g(y)\,dy} = e^{-3\int \frac{1}{y}dy} = e^{-3\log y}$$

$$= \frac{1}{y^3}$$

$\therefore$ (1) $\Rightarrow$

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + \frac{2y}{y} - \frac{4x}{y^3}\right)dy = 0$$

which is exact.

$\therefore$ Soln. is

$$\int_{y\ \text{constt.}} \left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c$$

$$\left(y + \frac{2}{y^2}\right)x + y^2 = c$$

Ans.

Ques. $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$ — (1)

Soln.

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$$

$$\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{(1) is not exact.}$$

Now

$$xy^2\,dx + 2x^2y^3\,dx + x^2y\,dy - x^3y^2\,dy = 0$$

$$xy\,(y\,dx + x\,dy) + x^2y^2(2y\,dx - x\,dy) = 0$$

Comparing with

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

$a=1$, $b=1$, $m=1$, $n=1$

$c=2$, $d=2$, $p=2$, $q=-1$

I.F. $= x^h y^k$

where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$

$\Rightarrow h-k=0$

$\frac{c+h+1}{p} = \frac{d+k+1}{q} \Rightarrow h+2k=-9$

Solving we get $h=-3$, $k=-3$

$\therefore$ I.F. $= \frac{1}{x^3 y^3}$

① $\Rightarrow$

$$\left(\frac{1}{x^2 y} + \frac{2}{x}\right) dx + \left(\frac{1}{x y^2} - \frac{1}{y}\right) dy = 0$$

which is exact.

$\therefore$ Soln. is

$$\int \left(\frac{1}{x^2 y} + \frac{2}{x}\right) dx + \int \frac{-1}{y}\, dy = c$$

y constt.

$$\frac{-1}{xy} + 2\log x - 2\log y = c$$

Ans.

Ques. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

I.F. $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1}{y}$

Ans – $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$

⊛ Leibnitz's Linear Eqn.

$$\frac{dy}{dx} + Py = Q$$, where P, Q are fns. of x only or may be constants.

Solution

$$I.F. = e^{\int P dx}$$

∴ Soln. is

$$\boxed{y(I.F.) = \int Q(I.F.)\, dx + c}$$

⟹ If $\frac{dx}{dy} + Px = Q$, P & Q are fns. of y only or constants

$$I.F = e^{\int P dy}$$

Soln is $x(I.F.) = \int Q(I.F.)\, dy + c$

Ques. $(1+x^3)\frac{dy}{dx} + 6x^2 y = 1+x^2$.

Soln. $\frac{dy}{dx} + \frac{6x^2}{1+x^3} y = \frac{1+x^2}{1+x^3}$.

$P = \frac{6x^2}{1+x^3}$, $Q = \frac{1+x^2}{1+x^3}$.

$$I.F. = e^{\int P dx} = e^{2\int \frac{3x^2}{1+x^3} dx} = e^{2\log(1+x^3)}$$

$$= (1+x^3)^2.$$

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Soln. is

$$y(I.F) = \int Q(I.F.)\, dx + c$$

$$y(1+x^3)^2 = \int \frac{1+x^2}{1+x^3}(1+x^3)^2\, dx + c$$

$$= \int (1+x^3+x^2+x^5)\, dx + c$$

$$y(1+x^3)^2 = x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C$$

Ans.

Ques. $\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$ if $y=0$ when $x = \frac{\pi}{2}$.

Soln. $P = \cot x$, $Q = 4x \operatorname{cosec} x$

$$I.F. = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x .$$

∴ Soln. is

$$y \sin x = 4\int x \operatorname{cosec} x \sin x\, dx + c$$

$$= 4\int x\, dx + c$$

$$y \sin x = 2x^2 + c \quad \text{—— (1)}$$

when $x = \frac{\pi}{2}$, $y = 0$

(1) $\Rightarrow$ $0 = \frac{\pi^2}{2} + c \Rightarrow c = -\pi^2/2$

$$\therefore\ y \sin x = 2x^2 - \frac{\pi^2}{2}$$

Ans.

Eqns. reducible to Linear form —
BERNOULLI'S EQN.

An eqn. of the form $\frac{dy}{dx} + Py = Qy^n$ — ①

where P & Q are fns. of x only or constants, is known as Bernoulli's eqn.

Divide both sides of ① by y^n.

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad \text{— ②}$$

Put $y^{1-n} = t \Rightarrow (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$$

$\therefore$ ② $\Rightarrow \frac{1}{1-n}\frac{dt}{dx} + Pt = Q$

$$\frac{dt}{dx} + P(1-n)\,t = Q(1-n)$$

which is linear & can be solved by method of Leibnitz eqn.

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Ques. $(x^3y^2 + xy)\,dx = dy$

Soln. $\frac{dy}{dx} = x^3y^2 + xy$

$\frac{dy}{dx} - xy = x^3y^2$

Dividing by y^2, we get

$\frac{1}{y^2}\frac{dy}{dx} - \frac{x}{y} = x^3$ ——— (1)

Put $y^{-1} = t \Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx}$

(1) $\Rightarrow -\frac{dt}{dx} - xt = x^3$

$\frac{dt}{dx} + tx = -x^3 \Rightarrow P = x,\; Q = x^3$

$I.F. = e^{\int x\,dx} = e^{x^2/2}$

$\therefore$ Soln. is

$t e^{x^2/2} = -\int x^3 e^{x^2/2}\,dx + c$

$\frac{e^{x^2/2}}{y} = -\int 2\log z\,dz + c$ $\qquad e^{x^2/2} = z$, $x e^{x^2/2}\,dx = dz$

$= -2(z\log z - z) + c$

$= -2\left(\frac{x^2}{2}e^{x^2/2} - e^{x^2/2}\right) + c$

$\frac{1}{y} = -x^2 + 2 + c e^{-x^2/2}$

Ans.

* Differential Eqns. of First Order + Higher Degree

Let $\frac{dy}{dx} = p$

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \text{-----} + P_n = 0 \quad \text{--- (1)}$$

is D.E. of first order and higher degree, where $P_1, P_2, \text{--}, P_n$ are fns. of x + y.

No. of arbitrary constts. = 1 = Order

* Equations Solvable for p

Resolve (1) into n linear factors

$$(p-f_1)(p-f_2) \text{-----} (p-f_n) = 0$$

$\Rightarrow p = f_1, \quad p = f_2, \text{-----}, p = f_n$

Solve these + we get

$f_1(x,y,c) = 0, \quad F_2(x,y,c) = 0, \text{---}, f_n(x,y,c) = 0$

$\therefore$ G.S. of (1) is

$$f_1 f_2 \text{----} f_n = 0.$$

Ques. $xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$

Soln. $xyp^2 - (x^2+y^2)p + xy = 0$

$$p = \frac{(x^2+y^2) \pm \sqrt{(x^2+y^2)^2 - 4x^2y^2}}{2xy}$$

$$= \frac{x^2+y^2 \pm (x^2-y^2)}{2xy}$$

$$\therefore p = \frac{x}{y}, \frac{y}{x}$$

$p = \frac{x}{y}$ and $p = y/x$

$\frac{dy}{dx} = \frac{x}{y}$

$y\,dy = x\,dx$

$y^2 - x^2 = c$

$y^2 - x^2 - c = 0$

$\frac{dy}{dx} = \frac{y}{x}$

$\frac{dy}{y} = \frac{dx}{x}$

$\log y = \log x + \log c$

$y = cx$

$y - cx = 0$

$\therefore$ General soln. is

$(y^2 - x^2 - c)(y - cx) = 0$

Ans.

Ques. $x^2p^2 + 3xyp + 2y^2 = 0$

Soln. $p = \frac{-3xy \pm \sqrt{9x^2y^2 - 8y^2x^2}}{2x^2}$

$= \frac{-3xy \pm xy}{2x^2}$

$\frac{-4xy}{2x^2}$

$p = -\frac{y}{x}$

$\frac{dy}{dx} + \frac{y}{x} = 0$

$\frac{dy}{y} + \frac{dx}{x} = 0$

$xy - c = 0$

$p = -2\frac{y}{x}$

$\frac{dy}{dx} + \frac{2y}{x} = 0$

$\frac{dy}{y} + 2\frac{dx}{x} = 0$

$x^2y - c = 0$

$\therefore$ General soln. is

$(xy - c)(x^2y - c) = 0$

Ans.

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Ques. $p(p+y) = x(x+y)$

Soln. $p^2 + py - (x^2 + xy) = 0$

$$p = \frac{-y \pm \sqrt{y^2 + 4x^2 + 4xy}}{2}$$

$$= \frac{-y \pm (2x+y)}{2}$$

$p = x$

$\frac{dy}{dx} = x$

$dy = x\,dx$

$y = \frac{x^2}{2} + C$

$y - \frac{x^2}{2} - c = 0$

$p = -x - y$

$\frac{dy}{dx} + y = -x$

$I.F. = e^{\int dx} = e^x$

$ye^x = -\int x e^x dx + c$

$ye^x = -(xe^x - e^x) + c$

$y + x - 1 - ce^{-x} = 0$

$\therefore$ General soln. is

$$\left(y - \frac{x^2}{2} \cancel{+} c\right)\left(y + x - 1 - ce^{-x}\right) = 0$$

Ans.

Ques. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Soln. $p - \frac{1}{p} = \frac{x^2 - y^2}{xy} = \frac{p^2 - 1}{p}$

$xyp^2 - (x^2 - y^2)p - xy = 0$

$p = \frac{x}{y}, -\frac{y}{x}$

$\Rightarrow \left(\frac{y^2}{2} - \frac{x^2}{2} - C\right)(xy - c) = 0$

Ans.

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⊛ Equations Solvable for y

If ① is solvable for y, then $y = f(x, p)$ — ②

$$\frac{dy}{dx} = p = F\left(x, p, \frac{dp}{dx}\right) \quad \text{—— ③}$$

③ is D.E. in p and x.

Let soln. of ② is $\phi(x, p, c) = 0$ —— ④

Eliminate p from ② & ④ which gives the reqd. soln.

If p can't be eliminated, then solve ② & ④ for x & y to get $x = \phi_1(p, c)$, $y = \phi_2(p, c)$.

These two constitute the soln. of ①.

⊛ Equations Solvable for x

If ① is solvable for x, then $x = f(y, p)$ — ②

$$\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \quad \text{—— ③}$$

Let soln. of ③ is $\phi(y, p, c) = 0$ —— ④

Eliminate p from ② & ④ which gives the reqd. soln.

If p can't be eliminated, then solve ② & ④ for x & y to get $y = \phi_1(p, c)$, $y = \phi_2(p, c)$.

These two constitute the soln. of ①.

Ques. $xp^2 - 2yp + ax = 0$. —— (1)

Soln. $$y = \frac{xp^2 + ax}{2p} = \frac{xp}{2} + \frac{ax}{2p}$$

$$\frac{dy}{dx} = p = \frac{p}{2} + \frac{x}{2}\frac{dp}{dx} + \frac{a}{2p} - \frac{ax}{2p^2}\frac{dp}{dx}$$

$$\frac{p}{2} - \frac{x}{2}\frac{dp}{dx} - \frac{a}{2p} + \frac{ax}{2p^2}\frac{dp}{dx} = 0$$

$$\frac{p}{2} - \frac{a}{2p} - \frac{x}{2}\frac{dp}{dx}\left(1 - \frac{a}{p^2}\right) = 0$$

$$\frac{p}{2}\left(1 - \frac{a}{p^2}\right) - \frac{x}{2}\frac{dp}{dx}\left(1 - \frac{a}{p^2}\right) = 0$$

$$\left(1 - \frac{a}{p^2}\right)\left(\frac{p}{2} - \frac{x}{2}\frac{dp}{dx}\right) = 0$$

Discarding $\left(1 - \frac{a}{p^2}\right)$ factor, we get

$$\frac{p}{2} - \frac{x}{2}\frac{dp}{dx} = 0$$

$$p = x\frac{dp}{dx} \Rightarrow \frac{dx}{x} = \frac{dp}{p}$$

$$\log x = \log p + \log c$$

$$x = pc \Rightarrow p = \frac{x}{c}$$

$$\therefore (1) \Rightarrow x\frac{x^2}{c^2} - 2y\frac{x}{c} + ax = 0$$

$$2y\frac{x}{c} = \frac{x^3}{c^2} + ax$$

$$2y = \frac{x^2}{c} + ac$$

Ans.

Ques. $y - 2px = \tan^{-1}(xp^2)$ —— ①

Soln. $y = 2px + \tan^{-1}(xp^2)$

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + \frac{p^2 + 2px\frac{dp}{dx}}{1+x^2p^4}$$

$$p + 2x\frac{dp}{dx} + \frac{p^2}{1+x^2p^4} + \frac{2px}{1+x^2p^4}\frac{dp}{dx} = 0$$

$$p + 2x\frac{dp}{dx} + \frac{p}{1+x^2p^4}\left(p + 2x\frac{dp}{dx}\right) = 0$$

$$\left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1+x^2p^4}\right) = 0$$

Neglecting $\left(1 + \frac{p}{1+x^2p^4}\right)$ factor, we get

$$p + 2x\frac{dp}{dx} = 0 \Rightarrow \frac{dx}{x} + 2\frac{dp}{p} = 0$$

$$\Rightarrow \log x + 2\log p = \log c$$

$$\Rightarrow xp^2 = c$$

$$\Rightarrow p = \sqrt{\frac{c}{x}}$$

Put in ①

$$y = 2\sqrt{\frac{c}{x}}\,x + \tan^{-1} c$$

$$y = 2\sqrt{cx} + \tan^{-1} c$$

Ans.

Ques. $3x^4p^2 - px - y = 0$

Solu. $y = 3x^4p^2 - px \quad \text{——} \quad (1)$

$$p = 12x^3p^2 + 6x^4p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$12x^3p^2 + 6x^4p\frac{dp}{dx} - 2p - x\frac{dp}{dx} = 0$$

$$6x^3p\left(2p + x\frac{dp}{dx}\right) - \left(2p + x\frac{dp}{dx}\right) = 0$$

$$\left(6x^3p - 1\right)\left(2p + x\frac{dp}{dx}\right) = 0$$

Neglecting $(6x^3p - 1)$ factor, we have

$$2p + x\frac{dp}{dx} = 0$$

$$2\frac{dx}{x} + \frac{dp}{p} = 0$$

$$2\log x + \log p = \log c \Rightarrow px^2 = c$$

$$p = c/x^2$$

Put in (1)

$$y = 3x^4\frac{c^2}{x^4} - \frac{c}{x^2}x$$

$$y = 3c^2 - \frac{c}{x}$$

Ans.

Ques. $p = \tan\left(x - \frac{p}{1+p^2}\right)$.

Soln. $$x = \tan^{-1} p + \frac{p}{1+p^2} \quad \text{——— (1)}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{1-p^2}{(1+p^2)^2}\frac{dp}{dy}$$

$$\cancel{\frac{1-p^2}{(1+p^2)^2}\frac{dp}{dy} = \frac{1}{p} - \frac{1}{1-p^2}}$$

$$\cancel{= \frac{1-p^2-}{p(1-p^2)}}$$

$$\frac{1}{p} = \frac{1+\cancel{p^2}+1-\cancel{p^2}}{(1+p^2)^2}\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$dy = \frac{2p}{(1+p^2)^2}\,dp \qquad 1+p^2 = t, \quad 2p\,dp = dt, \quad \int \frac{dt}{t^2}$$

$$y = \frac{-1}{1+p^2} + c$$

$$\Rightarrow y + \frac{1}{1+p^2} = c \quad \text{——— (2)}$$

Eqns. (1) and (2) together constitute the general soln.

Clairaut's Eqn.

An equation of the form $y = px + f(p)$ —— ①
is known as Clairaut's Eqn.

$$① \Rightarrow \frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0$$

Rejecting the factor $(x + f'(p))$, we get

$$\frac{dp}{dx} = 0 \Rightarrow p = c$$

$\therefore$ ① $\Rightarrow$ $y = cx + f(c)$ is the reqd. soln.

⟶ Soln. of Clairaut's eqn. is obtained by writing c for p.

Ques. $p = \log(px - y)$

Soln. $e^p = px - y$

$$\Rightarrow y = px - e^p$$

which is of Clairaut's form.

$\therefore$ Soln. is

$$y = cx - e^c$$

Ans.

Ques. $e^{3x}(p-1) + p^3 e^{2y} = 0$ —— ①

⟶ In problems having $e^{lx} + e^{my}$

Put $X = e^{kx}$, $Y = e^{ky}$

where $k = \text{HCF}(l, m)$.

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Put $X = e^x$, $Y = e^y$

$dX = e^x\,dx$, $dY = e^y\,dy$

$\therefore \dfrac{dY}{dX} = \dfrac{e^y}{e^x}\,p \Rightarrow P = \dfrac{Y}{X}\,p$

$\Rightarrow p = \dfrac{X}{Y}\,P$

$\therefore$ ① $\Rightarrow$

$$X^3\left(\frac{X}{Y}P - 1\right) + \frac{X^3}{Y^3}P^3Y^2 = 0$$

$$\frac{X^4P}{Y} - X^3 + \frac{X^3}{Y}P^3 = 0$$

$$\frac{X^3}{Y}\left(PX - Y + P^3\right) = 0$$

$$PX - Y + P^3 = 0$$

$$Y = PX + P^3$$

which is of Clairaut's form

$\therefore$ Soln. is

$$Y = CX + C^3$$

$$e^y = Ce^x + C^3$$

Ans.

Ques. $x^2(y - px) = yp^2$. — ①

Soln. Put $x^2 = X$, $y^2 = Y$

$2x\,dx = dX$, $2y\,dy = dY$

$P = \dfrac{dY}{dX} = \dfrac{y}{x}\dfrac{dy}{dx} = \sqrt{\dfrac{Y}{X}}\,p \Rightarrow p = \sqrt{\dfrac{X}{Y}}\,P$

① $\Rightarrow X\left(\sqrt{Y} - \sqrt{\dfrac{X}{Y}}\,P\,\sqrt{X}\right) = \sqrt{Y}\,\dfrac{X}{Y}\,P^2$

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$$\Rightarrow \sqrt{Y} - \frac{X}{\sqrt{Y}} P = \frac{\sqrt{Y}}{Y} P^2$$

$$\Rightarrow \sqrt{Y} - \frac{X}{\sqrt{Y}} P = \frac{P^2}{\sqrt{Y}}$$

$$\Rightarrow Y - PX = P^2$$

$$\Rightarrow Y = PX + P^2$$

which is of Clairaut's form

$\therefore$ Soln. is

$$Y = CX + C^2$$

$$y^2 = Cx^2 + C^2$$

Ans.

Ques. $p^2(x^2-1) - 2pxy + y^2 - 1 = 0$ — (1)

Soln. Put $x^2 = X$, $y^2 = Y$

$$2x\,dx = dX, \quad 2y\,dy = dY$$

$$P = \frac{dY}{dX} = \frac{y\,dy}{x\,dx} = \sqrt{\frac{Y}{X}}\,p \Rightarrow p = \sqrt{\frac{X}{Y}}\,P$$

$$(1) \Rightarrow \frac{X}{Y}P^2(X-1) - 2\sqrt{\frac{X}{Y}}\,P\sqrt{XY} + Y - 1 = 0$$

$$\frac{X^2P^2}{Y} - \frac{P^2X}{Y} - 2PX + Y - 1 = 0$$

Soln. $p^2x^2 - p^2 - 2pxy + y^2 - 1 = 0$

$$p^2x^2 - 2pxy + y^2 = 1 + p^2$$

$$(y - px)^2 = 1 + p^2$$

$$y - px = \sqrt{1 + p^2}$$

which is of clairaut's form

$$y - cx = \sqrt{1 + p^2}$$

Ans.