INFINITE SERIES

Sequence
A seque is a fnx. where domain is set N of all natural nos. whereas the range may be any set S.

$$f : N \longrightarrow S.$$

Real Sequence
Fnx. $f : N \longrightarrow \mathbb{R}$

Range of a Sequence
Set of all distinct terms of a sequence.

$\{x_n\} = \{(-1)^n\}$, Range $= \{-1, 1\}$.

Bounded Sequence
Bdd. above – Seq. $\{a_n\}$ is s.t.b. bdd. above if $\exists\, K \in \mathbb{R}$ s.t. $\boxed{a_n \leq K}$, $\forall\, n \in N$

Bdd. below – Seq. $\{a_n\}$ is bdd. below if $\exists\, k \in \mathbb{R}$ s.t. $\boxed{a_n \geq k}$, $\forall\, n \in N$.

Bdd. seq. $\quad k \leq a_n \leq K$

e.g. $\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, ----\right\}$, $\forall\, n \in \mathbb{N}$

$0 < a_n \leq 1 \Rightarrow$ bdd. seq.

$\{a_n\} = \{2^{n-1}\} = \{1, 2, 2^2, ----\}$

$a_n \geq 1 \Rightarrow$ bdd. below.

Convergent Seq.

$\lim_{a_n \to \infty} a_n$ is finite

e.g. $\left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots \right\}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$, finite.

Divergent Seq.

$\lim_{n\to\infty} a_n = +\infty$ or $-\infty$

e.g. $\{n^2\}$ and $\{-2^n\}$.

Infinite Series

If $\{u_n\}$ is a sequence of real nos., then the expression $u_1 + u_2 + \ldots + u_n + \ldots$ (i.e. the sum of terms of the seq., which are infinite in no.) is called infinite series. Denoted by Σu_n.

Positive Term Series

If Σu_n has all $u_n > 0$, $\forall n$.

Alternating Series

Terms are alternate +ve or −ve.

i.e. $\Sigma (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$,

$u_n > 0$, $\forall n$

Partial Sums

S_n is partial sum of Σu_n

i.e. $S_n = u_1 + u_2 + \ldots + u_n$.

(*) Behaviour of Infinite Series

Σu_n behaves as $\{S_n\}$ sequence behaves

→ $\lim_{n\to\infty} S_n$ is finite, then Σu_n cgs.

→ $\lim_{n\to\infty} S_n = \pm\infty$, then Σu_n dgs.

→ $\{S_n\}$ is bdd. & neither cgs. nor dgs., then Σu_n oscillates finitely

→ $\{S_n\}$ is unbdd. & neither cgs. nor dgs., then Σu_n oscillates infinitely.

(*) Nature of Geometric Series $1 + x + x^2 + x^3 + \cdots$

→ cgs. if $-1 < x < 1$ i.e. $|x| < 1$

→ dgs if $x \geq 1$

→ oscillates finitely if $x = -1$

→ oscillates infinitely if $x < -1$.

(*) Necessary Condition for cgs. of a +ve term series

If a +ve term series Σu_n is cgt., then $\lim_{n\to\infty} u_n = 0$.

Pf:- Let S_n be n^{th} partial sum of Σu_n.

Since Σu_n is cgt.

$\Rightarrow \{S_n\}$ is cgt.

$\Rightarrow \lim_{n\to\infty} S_n = l$

Also $\lim_{n\to\infty} S_{n-1} = l$

Now $S_n - S_{n-1} = u_n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= l - l$$

$$= 0$$

$\therefore$ $\boxed{\Sigma u_n \text{ is cgt} \Rightarrow \lim_{n\to\infty} u_n = 0}$.

→ Converse not true always.

e.g. Series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

But series is a dgt. series. By p-test $p=1$ dgt.

$\Rightarrow \lim_{n\to\infty} u_n = 0 \Rightarrow \Sigma u_n$ may or may not be cgt.

→ $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \Sigma u_n$ is dgt.

(*) A +ve term series either cgs. or dgs. to $+\infty$.

(*) Necessary & sufficient condition for the cgs. of a +ve term series Σu_n is that $\{S_n\}$ is bdd. above.

(*) Cauchy's General Principle of Cgs. of Series.

Σu_n is cgt. iff given $\epsilon > 0$, however small $\exists$ +ve integer p s.t.

$|S_{n+p} - S_n| < \epsilon \quad \forall\, n \geq m, \quad m, p \in \mathbb{N}$.

$S_{m+p} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} + \frac{1}{m+p1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p}$

$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$

Ques. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge (by applying CGP of cgs).

Soln. If possible, let $\sum \frac{1}{n}$ is cgt.

By CGP of cgs.

$|S_{n+p} - S_n| < \epsilon \quad \forall\, n \geq m, \; p \in \mathbb{N}$

Take $\epsilon = \frac{1}{2}$ $\therefore |S_{n+p} - S_n| < \frac{1}{2} \quad \forall\, n \geq m, \; p \in \mathbb{N}$

Put $n = m$

$$|S_{m+p} - S_m| < \frac{1}{2}, \quad \forall\, p \in \mathbb{N}$$

i.e. $\left| \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p} \right| < \frac{1}{2} \quad \forall\, p \in \mathbb{N}$

or $\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+p} < \frac{1}{2}$

Put $p = m$

$$\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} < \frac{1}{2} \quad \text{--- (1)}$$

But

$$\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} > \frac{1}{m+m} + \frac{1}{m+m} + \cdots + \frac{1}{2m}$$

$$= \frac{m}{2m} = \frac{1}{2}$$

$$\therefore \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} > \frac{1}{2} \quad \text{--- (2)}$$

(2) contradicts (1)

$\therefore$ Our supposition is wrong.

$\Rightarrow \sum \frac{1}{n}$ does not converge.

Comparison Tests

Let $\sum u_n$ and $\sum v_n$ be two +ve term series.

I (a) $u_n \le k v_n$, $\forall\ n \ge m$

If $\sum v_n$ is cgt., then $\sum u_n$ is cgt.

I (b) $u_n > k v_n$ $\forall\ n > m$

If $\sum v_n$ is dgt., then $\sum u_n$ is dgt.

II $H < \frac{u_n}{v_n} < K$ (H, K are independent of n)

Then $\sum u_n$, $\sum v_n$ converge or diverge together

III $\boxed{\text{Limit Comparison Test}}$

→ $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, then $\sum u_n$ & $\sum v_n$ converge or diverge together.

→ $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ and $\sum v_n$ cgs., then $\sum u_n$ cgs.

→ $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ & $\sum v_n$ dgs., then $\sum u_n$ dgs.

→ $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ & $\sum u_n$ cgs., then $\sum v_n$ cgs.

IV (a) If $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$, $\forall\ n > m$ & $\sum v_n$ is cgt. then $\sum u_n$ is cgt.

(b) If $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$ $\forall\ n > m$ & $\sum v_n$ is dgt. then $\sum u_n$ is dgt.

⊛ p-Series Test

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

cgs. if $p > 1$
dgs. if $p \leq 1$.

Ques. Examine the cgs. of the series :

$$1 + \frac{1}{4^{2/3}} + \frac{1}{9^{2/3}} + \frac{1}{16^{2/3}} + \cdots$$

Soln.
$$\frac{1}{1^{4/3}} + \frac{1}{2^{4/3}} + \frac{1}{3^{4/3}} + \frac{1}{4^{4/3}} + \cdots$$

$$= \sum \frac{1}{n^{4/3}}$$

$p = \frac{4}{3} > 1 \Rightarrow$ By p-series test, given series converges.

Ques. Test the cgs. of following series :

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \cdots$$

Soln. $u_n = \frac{n^n}{(n+1)^{n+1}}$ leaving first term.

$$= \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$$

Take $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{1}{1+\frac{1}{n}} = \frac{1}{e}$$

which is finite & non-zero.

$\therefore$ Σu_n & Σv_n converge or diverge together.

By p-test, $\Sigma v_n = \Sigma \frac{1}{n}$ dgs.

Thus, Σu_n also diverges.

Ques. $\sum \left(\sqrt[3]{n^3+1} - n \right)$.

Solu. $u_n = \sqrt[3]{n^3+1} - n$

$$= n\left[\sqrt[3]{1+\frac{1}{n^3}} - 1\right]$$

$$u_n = n\left(1 + \frac{1}{3}\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{\underline{|2}}\left(\frac{1}{n^3}\right)^2 + \cdots - 1\right)$$

$$= n\left(\frac{1}{3}\frac{1}{n^3} - \frac{1}{9}\frac{1}{n^6} + \cdots\right)$$

$$= \frac{1}{3}\frac{1}{n^2} - \frac{1}{9}\frac{1}{n^5} + \cdots$$

$$= \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9}\frac{1}{n^3} + \cdots\right]$$

Take $v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$ which is finite & non-zero

$\therefore$ Σu_n & Σv_n cgs. or dgs. together.

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$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\sum v_n = \sum \frac{1}{n^2}$ is cgt. $\because p = 2 > 1$.

$\Rightarrow \sum u_n$ is also cgt.

Ques. $\sum\left(\frac{1}{n} - \log\frac{n+1}{n}\right)$

Solu. $u_n = \frac{1}{n} - \log\frac{n+1}{n}$

$= \frac{1}{n} - \log\left(1+\frac{1}{n}\right)$

$u_n = \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right]$

$= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \cdots$

Let $v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \cdots\right]$

$= \frac{1}{2}$, which is finite & non-zero

$\therefore \sum u_n$ & $\sum v_n$ behave alike.

$\sum v_n = \sum \frac{1}{n^2}$, $p = 2 > 1$

$\therefore$ By p-series test, $\sum v_n$ cgs.

$\Rightarrow \sum u_n$ converges.

Ques $\frac{1}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{3}{5\cdot 7} + \cdots$

Solu. $u_n = \frac{n}{(2n-1)(2n+1)}$

$1+2+3+\cdots$

$1+(n-1)1 = n$

$1\cdot 3\cdot 5 \cdots$

$1+(n-1)2 = 2n-1$

$3\cdot 5\cdot 7 \cdots$

$3+(n-1)2 = 2n+1$

$$u_n = \frac{n}{4n^2\left(1-\frac{1}{2n}\right)\left(1+\frac{1}{2n}\right)}$$

$$= \frac{1}{4n\left(1-\frac{1}{2n}\right)\left(1+\frac{1}{2n}\right)}$$

Let $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{4\left(1-\frac{1}{2n}\right)\left(1+\frac{1}{2n}\right)} = \frac{1}{4}$$

which is finite & non-zero.

$\therefore$ Σu_n & Σv_n behave alike.

$\Sigma v_n = \Sigma \frac{1}{n}$, $p = 1$

$\therefore$ By p-series test, Σv_n is dgt.

$\Rightarrow \Sigma u_n$ is dgt.

(*) D'Alembert's Ratio Test

If Σu_n is a +ve term series and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$,

then (i) Σu_n is cgt. if $l > 1$

(ii) Σu_n is dgt. if $l < 1$

(iii) Test fails if $l = 1$

Ques: $\frac{1^2 . 2^2}{\lfloor 1} + \frac{2^2 . 3^2}{\lfloor 2} + \frac{3^2 . 4^2}{\lfloor 3} + \frac{4^2 . 5^2}{\lfloor 4} + \text{------}$

Solu. Here $u_n = \frac{n^2 (n+1)^2}{\lfloor n}$

$$u_{n+1} = \frac{(n+1)^2 (n+2)^2}{\lfloor n+1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2 \cancel{(n+1)^2}}{\lfloor n} \times \frac{\lfloor n+1}{\cancel{(n+1)^2} (n+2)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)\, n^2}{(n+2)^2}$$

$$= \lim_{n\to\infty} \frac{n^3 + n^2}{(n+2)^2}$$

$$= \lim_{n\to\infty} \frac{n^3 \left(1 + \frac{1}{n}\right)}{n^2 \left(1 + \frac{2}{n}\right)^2}$$

$$= \infty > 1$$

$\therefore$ Σu_n is cgt. by Ratio Test.

Ques. $\frac{2}{1} + \frac{2\cdot5\cdot8}{1\cdot5\cdot9} + \frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13} + \text{------}$

Solu. $u_n = \frac{2\cdot5\cdot8\cdot\text{ ----- }(3n-1)}{1\cdot5\cdot9\cdot\text{ ---- }(4n-3)}$

$u_{n+1} = \frac{2\cdot5\cdot8\cdot\text{ ----- }(3n-1)\,(3n+2)}{1\cdot5\cdot9\cdot\text{ ----- }(4n-3)\,(4n+1)}$

$$\underset{n\to\infty}{lt} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{lt} \frac{4n+1}{3n+2}$$

$$= \underset{n\to\infty}{lt} \left(\frac{4+\frac{1}{n}}{3+\frac{2}{n}}\right) = \frac{4}{3} > 1$$

$\therefore$ Σu_n is cgt.

Ques. $\sum \frac{\lfloor n \; 2^n}{n^n}$.

Solu. $u_n = \frac{\lfloor n \; 2^n}{n^n}$, $u_{n+1} = \frac{\lfloor n+1 \; 2^{n+1}}{(n+1)^{n+1}}$

$$\underset{n\to\infty}{lt} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{lt} \frac{\lfloor n \; 2^n}{n^n} \times \frac{(n+1)^{n+1}}{\lfloor n+1 \; 2^{n+1}}$$

$$= \underset{n\to\infty}{lt} \frac{1}{2} \frac{(n+1)^n}{n^n}$$

$$= \frac{1}{2} \underset{n\to\infty}{lt} \left(1+\frac{1}{n}\right)^n = \frac{e}{2} >$$

$2 < e < 3 \Rightarrow 1 < \frac{e}{2} < \frac{3}{2}$

$\therefore$ Σu_n is cgt.

Ques. $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \text{-----}$

Soln: $u_n = \frac{x^n}{(n+1)\sqrt{n+2}}$, $u_{n+1} = \frac{x^{n+1}}{(n+2)\sqrt{n+3}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{(n+1)\sqrt{n+2}} \times \frac{(n+2)\sqrt{n+3}}{x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1} \sqrt{\frac{n+3}{n+2}} \cdot \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \sqrt{\frac{1+\frac{3}{n}}{1+\frac{2}{n}}} \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

Σu_n cgs. for $\frac{1}{x} > 1 \Rightarrow x < 1$

dgs. for $\frac{1}{x} < 1 \Rightarrow x > 1$.

When $x = 1$. $u_n = \frac{1}{(n+1)\sqrt{n+2}}$

$\lim_{n\to\infty} u_n = 0$

As Σu_n is a +ve term series & $\lim_{n\to\infty} u_n = 0$.

$\therefore \Sigma u_n$ converges.

Hence Σu_n cgs. for $x \le 1$
dgs. for $x > 1$. Ans.

Que: $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \cdots + \frac{2^{n+1}-2}{2^{n+1}+1}x^n + \cdots$ $(x >$

Solu: $u_n = \frac{2^{n+1}-2}{2^{n+1}+1}x^n$

$u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1}x^{n+1}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2^{n+1}-2}{2^{n+1}+1} \cdot \frac{2^{n+2}+1}{2^{n+2}-2} \cdot \frac{1}{x}$

$= \frac{1}{x} \lim_{n\to\infty} \frac{1 - \frac{2}{2^{n+1}}}{1 + \frac{1}{2^{n+1}}} \times \frac{1 + \frac{1}{2^{n+2}}}{1 - \frac{2}{2^{n+2}}}$

$= \frac{1}{x}$

Σu_n cgs. if $\frac{1}{x} > 1 \Rightarrow x < 1$
dgs. if $\frac{1}{x} < 1 \Rightarrow x > 1$

When $\underline{x = 1}$, $\lim_{n\to\infty} \frac{2^{n+1}-2}{2^{n+1}+1} = 1 \neq 0$

Σu_n does not converge.
$\therefore$ Being a +ve term series, Σu_n dgs

Hence Σu_n : cgs. for $x < 1$
dgs. for $x \geq 1$ $\Big\}$ Ans.

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⊛ Raabe's Test

If Σu_n is a +ve term series and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then

(i) Σu_n is cgt. if $l > 1$

(ii) Σu_n is dgt. if $l < 1$

(iii) Test fails if $l = 1$.

$\longrightarrow$ This test is used when Ratio test fails & when in ratio test $\frac{u_n}{u_{n+1}}$ does not contain no. e.

⊛ Logarithmic Test

Σu_n is +ve term series and

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l.$$

(i) cgs. if $l > 1$

(ii) dgs. if $l < 1$

(iii) Test fails if $l = 1$.

$\longrightarrow$ This test is used after failure of Ratio test & when $\frac{u_n}{u_{n+1}}$ involves e.

⊛ Gauss Test

Σu_n is series of +ve terms and

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

then Σu_n cgs. if $\lambda > 1$

dgs. if $\lambda \leq 1$

where $O\left(\frac{1}{n^2}\right)$ has terms of $\frac{1}{n^2}$ & its higher powers.

Ques: $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

Solu: $u_n = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}$

$u_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n+2)^2}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+1)^2} = 1$$

$\therefore$ Ratio test fails.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{4n^2+4+8n-4n^2-1-4n}{(2n+1)^2}\right]$$

$$= n\left[\frac{3+4n}{(1+2n)^2}\right]$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{3+4n}{(1+2n)^2}\right]$$

$$= 1$$

$\therefore$ Raabe's Test fails.

Now,

$$\frac{u_n}{u_{n+1}} = \frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2} = \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{1}{n^2}+\frac{2}{n}\right)\left(1-\frac{1}{n}+\frac{3}{4n^2} - \cdots\right)$$

$$= 1+\frac{1}{n}-\frac{1}{4n^2}+\cdots$$

$$= 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Here $\lambda = 1 \Rightarrow$ By Gauss Test, Σu_n dgs.

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$$\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{x/a}\right]^{a} = e^{a}$$

Ques. $1 + \frac{2x}{\lfloor 2} + \frac{3^2 x^2}{\lfloor 3} + \frac{4^3 x^3}{\lfloor 4} + \frac{5^4 x^4}{\lfloor 5} + \cdots$

Soln. $u_n = \frac{(n+1)^n x^n}{\lfloor n+1}$ $\qquad \frac{n^{n-1} x^{n-1}}{\lfloor n}$

$$u_{n+1} = \frac{(n+2)^{n+1} x^{n+1}}{\lfloor n+2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n \cdot \frac{1}{x}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2}\cdot\frac{1}{x}$$

$$= \frac{e}{e^2}\cdot\frac{1}{x} = \frac{1}{ex}$$

By Ratio test, Σu_n cgs. if $\frac{1}{ex} > 1 \Rightarrow x < \frac{1}{e}$

dgs if $\frac{1}{ex} < 1 \Rightarrow x > \frac{1}{e}$

test fails if $\frac{1}{ex} = 1 \Rightarrow x = \frac{1}{e}$

Now $x = \frac{1}{e}$, $\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\cdot e$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left[n\log\left(1+\frac{1}{n}\right) - n\log\left(1+\frac{2}{n}\right) + \log e\right]$$

$$= \lim_{n\to\infty} n\left[n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) - n\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3} - \cdots\right) + 1\right]$$

$$= \lim_{n\to\infty} n\left(\frac{3}{2n} - \frac{7}{3n^2} + \frac{15}{4n^3} - \cdots\right)$$

$$= \frac{3}{2} > 1$$

$\Rightarrow \sum u_n$ is cgt. for $x = \frac{1}{e}$.

Hence $\sum u_n$ is cgt. for $x \le \frac{1}{e}$
dgt. for $x > \frac{1}{e}$. Ans.

Ques. $$\sum \frac{\lfloor n}{x(x+1)(x+2)\cdots(x+n-1)}$$

Soln. $$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x+n}{n+1} = 1$$

$\therefore$ Ratio test fails.

We apply Gauss Test

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{x}{n}\right)\left(1+\frac{1}{n}\right)^{-1}$$

$$= \left(1+\frac{x}{n}\right)\left(1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \cdots\right)$$

$$= 1 + \frac{x}{n} - \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

$\sum u_n$ is cgt. if $x - 1 > 1 \Rightarrow x > 2$
dgt. if $x - 1 \le 1 \Rightarrow x \le 2$.

Ques. $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots\ldots \quad (x>0)$.

Solu. $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$ (Neglecting 1st term)

$u_{n+1} = \left(\frac{n+2}{n+3}\right)^{n+1} x^{n+1}$.

By Ratio test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)^n}{(n+2)^n} \times \frac{(n+3)^n (n+3)}{(n+2)^n (n+2)} \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2} \cdot \frac{\left[\left(1+\frac{3}{n}\right)^{n/3}\right]^3}{\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2} \cdot \frac{1+\frac{3}{n}}{1+\frac{2}{n}} \cdot \frac{1}{x}$$

$$= \frac{e}{e^2} \frac{e^3}{e^2} \cdot \frac{1}{x} = \frac{1}{x}$$

Σu_n cgs. if $\frac{1}{x} > 1 \Rightarrow x < 1$

dgs. if $\frac{1}{x} < 1 \Rightarrow x > 1$.

fails when $\frac{1}{x} = 1 \Rightarrow x = 1$

When $x=1$, $\lim_{n\to\infty} \frac{(n+1)^n}{(n+2)^n} = \frac{e}{e^2} = \frac{1}{e} \neq 0$

Since Σu_n is +ve term series & $\lim_{n\to\infty} u_n \neq 0$

$\Rightarrow \Sigma u_n$ is dgt for $x=1$

Hence,

Σu_n is cgt. for $x < 1$
dgt. for $x \geq 1$ } Ans.

Ques: $1 + \frac{3}{7}x + \frac{3 \cdot 6}{7 \cdot 10}x^2 + \frac{3 \cdot 6 \cdot 9}{7 \cdot 10 \cdot 13}x^3 + \cdots$

Solu: $u_n = \frac{3 \cdot 6 \cdot 9 \cdots (3n)}{7 \cdot 10 \cdot 13 \cdots (3n+4)} x^n$

$$\lt_{n\to\infty} \frac{u_n}{u_{n+1}} = \lt_{n\to\infty} \frac{3n+7}{3n+3} \cdot \frac{1}{x} = \frac{1}{x}$$

By ratio test, Σu_n cgs. for $x < 1$
dgs. for $x > 1$.

By Raabe's test, $x = 1$

$$\lt_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lt_{n\to\infty} \frac{4n}{3n+3}$$

$$= \frac{4}{3} > 1$$

$\therefore$ Σu_n is cgt. for $x = 1$.

Hence,

Σu_n cgs. for $x \le 1$
dgs. for $x > 1$ Ans.

Cauchy's Root Test

If Σu_n is a +ve term series & $\lim_{n\to\infty} u_n^{1/n} = l$

then (i) Σu_n cgs. if $l < 1$

(ii) Σu_n dgs. if $l > 1$

(iii) Test fails if $l = 1$.

Ques: Test the convergence of $\sum \frac{(n - \log n)^n}{2^n n^n}$.

Soln. $u_n = \frac{(n - \log n)^n}{2^n n^n}$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n - \log n}{2n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{2} - \frac{\log n}{n}\right)$$

$$= \frac{1}{2} - \lim_{n\to\infty} \frac{\log n}{n}$$

$\lim_{n\to\infty} \frac{\log n}{n} \left(\frac{\infty}{\infty}\right)$

$\lim_{n\to\infty} \frac{1/n}{1} = 0$

$$= \frac{1}{2} < 1$$

$\Rightarrow \Sigma u_n$ cgs. by root test

Ques. $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$

Soln. $u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$

$$\lim_{n\to\infty} u_n^{1/n} = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-1} = \frac{1}{e-1} < 1$$

$\therefore$ By Root test, Σu_n is cgt.

$2 < e < 3$

$1 < e-1 < 2 \Rightarrow \frac{1}{e-1} < 1$

Ques. Discuss the convergence of

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \infty.$$

Soln. $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$ (Neglecting first term)

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n+1}{n+2} . x = x$$

By Root test, Σu_n cgs. if $x < 1$
dgs. if $x > 1$
Test fails if $x = 1$.

Now $\underline{\underline{x = 1}}$

$$u_n = \left(\frac{n+1}{n+2}\right)^n$$

$$\lim_{n\to\infty} u_n = \frac{1}{e} \neq 0$$

Σu_n is +ve term series & $\lim_{n\to\infty} u_n \neq 0$.

$\Rightarrow$ Σu_n is dgt. for $x = 1$.

$\therefore$ Σu_n is cgt. for $x < 1$
dgt. for $x \geq 1$ } Ans.

monotonic ↓ $a_{n+1} - a_n \le 0$

$\Rightarrow a_{n+1} \le a_n$

⊛ Cauchy's Integral Test

If for $x \ge 1$, $f(x)$ is a non-negative monotonic ↓ fnx. of x s.t. $f(n) = u_n$, ∀ positive integral values of n, then the series $\sum u_n$ & integral $\int_1^\infty f(x)\,dx$ converge or diverge together.

$\Longrightarrow$ If $x \ge k$, then $\sum u_n$ & $\int_k^\infty f(x)\,dx$ converge or diverge together.

$\Longrightarrow$ If $\int_1^\infty f(x)\,dx$ cgs., then $\int_1^\infty f(x)\,dx$ = a fixed finite no.

$\Longrightarrow$ If $\int_1^\infty f(x)\,dx$ dgs., then $\int_1^\infty f(x)\,dx = +\infty$.

Ques Using Integral test, discuss convergence of $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$.

Soln. Let $u_n = \frac{1}{n\sqrt{n^2-1}} = f(n)$

$$f(x) = \frac{1}{x\sqrt{x^2-1}}$$

For $x \ge 2$, $f(x)$ is +ve & monotonic ↓

$$\frac{1}{2\sqrt{3}} > \frac{1}{3\sqrt{8}} > \frac{1}{4\sqrt{15}} > \cdots$$

∴ Cauchy's Integral test is applicable.

Now

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$$

Put $\sqrt{x^2-1} = t \Rightarrow x^2 = t^2+1 \Rightarrow x\,dx = t\,dt$

$\Rightarrow dx = \frac{t\,dt}{\sqrt{t^2+1}}$

$$\int_2^\infty f(x)\,dx = \int_{\sqrt{3}}^\infty \frac{t\,dt}{\sqrt{t^2+1}} \times \frac{1}{\sqrt{t^2+1}\cdot t}$$

$$= \int_{\sqrt{3}}^\infty \frac{dt}{t^2+1} = \left(\tan^{-1} t\right)_{\sqrt{3}}^\infty$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} = \text{finite}.$$

$\therefore \int_2^\infty f(x)\,dx$ is cgt.

$\Rightarrow \sum u_n$ also converges.

Ques Show that the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ & diverges if $0 < p \leq 1$.

Soln $u_n = \frac{1}{n^p} = f(n) \Rightarrow f(x) = \frac{1}{x^p}$

For $x \geq 1$, $f(x)$ is +ve & monotonic $\downarrow$.

$\therefore$ Cauchy's test is applicable.

Case I. $p \neq 1$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$$

Subcase 1: $p > 1 \Rightarrow p-1 > 0$

$$\int_1^\infty f(x)\,dx = -\frac{1}{p-1}\left(\frac{1}{x^{p-1}}\right)_1^\infty$$

$$= \frac{-1}{p-1}(0-1) = \frac{1}{p-1} = \text{finite}$$

$$\Rightarrow \int_1^\infty f(x)\,dx \text{ cgs.} \Rightarrow \Sigma u_n \text{ cgs.}$$

Subcase 2: $0 < p < 1 \Rightarrow p-1 < 0$

$$\Rightarrow 1-p > 0$$

$$\int_1^\infty f(x)\,dx = \frac{1}{1-p}\left(x^{1-p}\right)_1^\infty$$

$$= \frac{1}{1-p}(\infty - 1) = \infty$$

$$\Rightarrow \int_1^\infty f(x)\,dx \text{ dgs.} \Rightarrow \Sigma u_n \text{ dgs.}$$

Case II: $p = 1$ $\quad f(x) = \frac{1}{x}$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = (\log x)_1^\infty = \infty$$

$$\Rightarrow \int_1^\infty f(x)\,dx \text{ dgs} \Rightarrow \Sigma u_n \text{ dgs.}$$

$\therefore \Sigma u_n$ is cgt. for $p > 1$
dgt. for $0 < p \leq 1$ } Ans.

$a_{n+1} - a_n \leq 0$

Ques: Using Integral test, discuss cgs. of

$$\sum \frac{n}{(n^2+1)^2}$$

Soln: $u_n = \dfrac{n}{(n^2+1)^2} = f(n)$

$\Rightarrow \quad f(x) = \dfrac{x}{(x^2+1)^2}$.

For $x \geq 1$, $f(x)$ is +ve & monotonic $\downarrow$.

$\therefore$ Cauchy's Integral test is applicable.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x}{(x^2+1)^2}\,dx \qquad \begin{aligned} x^2+1 &= t \\ x\,dx &= \frac{dt}{2} \end{aligned}$$

$$= \int_2^\infty \frac{dt}{2t^2}$$

$$= \frac{-1}{2}\left(\frac{1}{t}\right)_2^\infty = \frac{-1}{2}\left(0 - \frac{1}{2}\right) = \frac{1}{4}$$

$\therefore \int_1^\infty f(x)\,dx$ cgs.

$\Rightarrow \sum u_n$ also converges.

Leibnitz's Test on Alternating Series

$\sum (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots \quad (u_n > 0)$

converges if

(i) $u_n > u_{n+1} \quad \forall\, n$

(ii) $\lim\limits_{n\to\infty} u_n = 0$.

$u_n = \frac{(-1)^n}{n^2+1} = \frac{1}{n^2+1}$, $u_{n+1} = \frac{1}{(n+1)^2-1}$

$\lim_{n\to\infty} u_n = 0$.

Ques. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$

Soln. $u_n = \frac{1}{n}$, $u_{n+1} = \frac{1}{n+1}$

$\frac{1}{n} > \frac{1}{n+1} \quad \forall n \Rightarrow u_n > u_{n+1} \quad \forall n$

Now $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Both the conditions are satisfied, thus $\Sigma(-1)^{n-1} u_n$ is cgt.

Ques. $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \cdots$

Soln. $u_n = \frac{1}{(n+1)^3}(1+2+3+\cdots+n)$

$u_{n+1} = \frac{1}{(n+2)^3}[1+2+3+\cdots+n+n+1]$

Now $u_n = \frac{1}{(n+1)^3}\frac{n(n+1)}{2} \Rightarrow u_n = \frac{n}{2(n+1)^2}$.

$u_{n+1} = \frac{n+1}{2(n+2)^2}$

$$u_n - u_{n+1} = \frac{n}{2(n+1)^2} - \frac{n+1}{2(n+2)^2}$$

$$= \frac{n(n^2+4n+4) - (n+1)(n^2+2n+1)}{2(n+1)^2(n+2)^2}$$

$$= \frac{n^2+n-1}{2(n+1)^2(n+2)^2} > 0 \quad \forall n$$

$\therefore u_n > u_{n+1} \quad \forall n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2(n+1)^2} = 0$$

Both conditions are satisfied

$\therefore$ Given series is cgt.

Ques: $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + ----$

Ques Test the convergence of the series

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + ----- \infty.$$

Also find the sum of the series.

Soln: $\left(\frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{9} - \frac{1}{4}\right) + \left(\frac{1}{27} - \frac{1}{8}\right) + ----$

Here $u_n = \frac{1}{3^n} - \frac{1}{2^n}$

$u_{n+1} = \frac{1}{3^{n+1}} - \frac{1}{2^{n+1}}$

$$u_n - u_{n+1} = \frac{1}{3^n} - \frac{1}{2^n} - \frac{1}{3^{n+1}} + \frac{1}{2^{n+1}}$$

$$= \frac{2}{3^{n+1}} - \frac{1}{2^{n+1}}$$

$$= \frac{2^{n+2} - 3^{n+1}}{6^{n+1}} < 0 \quad \forall n$$

$\Rightarrow u_n < u_{n+1} \quad \forall n$

first condition of Leibnitz test is not satisfied.

$\lim_{n\to\infty} u_n = 0$ is satisfied.

Now

$$\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$$

$$= \sum \frac{1}{3^n} - \sum \frac{1}{2^n}$$

$$= \sum u_n - \sum v_n$$

Now $\sum u_n$ is cgt.

$\because \lim_{n\to\infty} u_n^{1/n} = \frac{1}{3} < 1$

Also $\sum v_n$ is cgt. $\because \lim_{n\to\infty} v_n^{1/n} = \frac{1}{2} < 1$

$\therefore$ Given series is also cgt.

Sum of the series

$$= \left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$$

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$= \frac{1}{2} - 1 = \frac{-1}{2}$$

Ans.

Absolute Convergence of a Series

If a convergent series whose terms are not all +ve, remains convergent when all its terms are made +ve, then it is called an absolutely cgt series i.e.

The series $\sum u_n$ is s.t.b. abs. cgt.

if $\sum |u_n|$ is convergent series.

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④ **Conditionally Convergent Series**

A series is s.t.b. conditionally cgt if it is cgt but does not converge absolutely.

Ques. Test whether the series is abs. cgt or conditionally cgt. $\sum \frac{(-1)^{n-1}}{2n-1}$.

Solu. $u_n = \frac{1}{2n-1}$, $u_{n+1} = \frac{1}{2n+1}$

$u_n > u_{n+1} \quad \forall\, n$

$\lim_{n\to\infty} u_n = 0 \Rightarrow$ By Leibnitz's test, the series is cgt.

$\sum |u_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

$u_n = \frac{1}{2n-1}$ Take $v_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0$

By Comparison test, $\sum u_n$ & $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{n}$ is dgt. series by p-series test $(p=1)$

$\therefore \sum |u_n|$ is also dgt.

$\therefore$ Given series is conditionally cgt.

Ques. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - \cdots$

Soln. $u_n > u_{n+1}$ & $\lim_{n\to\infty} u_n = 0$

$\Rightarrow$ By Leibnitz's test, $\sum u_n$ cgs.

$\Sigma |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots = \Sigma \frac{1}{n^2}$

By p-series test, $p = 2 > 1$.
$\therefore$ $\Sigma |u_n|$ is cgt.
$\therefore$ Given series is absolutely cgt.

(*) Every Absolutely convergent series is convergent.
i.e. If $\Sigma |u_n|$ is cgt, then Σu_n is cgt.

NOTE $\Longrightarrow$ $AC \Longrightarrow C$
But $C \not\Longrightarrow AC$

e.g. $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\cdots$

$u_n > u_{n+1}$ & $\lim_{n\to\infty} u_n = 0$
$\therefore$ By Leibnitz test, given series is cgt.

But $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$

$p=1$, By p-series test, this is dgt.

NOTE $\Longrightarrow$ $\Sigma |u_n|$ is dgt. $\not\Longrightarrow$ Σu_n is dgt.

(*) <u>Power Series</u>
A series of the form $\sum_{n=0}^{\infty} a_n x^n$, where a_i's are independent of x, is called a power series in x.

(1) Interval of Convergence

The interval $(-l, l)$ is called I.O.C.

The power series cgs. in $(-l, l)$ & dgs. outside $(-l, l)$ }

→ Since a_i's can be +ve or -ve for cgs. of $\sum a_n x^n$, we test $\sum |a_n x^n|$

Hence AC ⇒ C

$\sum a_n x^n$ cgs. is checked.

Ques: For what value of x does the series $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ converges absolutely?

Soln. $\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} (-1)^n (4x+1)^n$.

$$|u_n| = |(4x+1)^n|$$

$$|u_{n+1}| = |(4x+1)^{n+1}|$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{|(4x+1)^n|}{|(4x+1)^{n+1}|} = \frac{1}{|4x+1|}$$

∴ $\sum |u_n|$ cgs. if $\frac{1}{|4x+1|} > 1$ By Ratio test

$$\Rightarrow |4x+1| < 1$$

$$\Rightarrow -1 < 4x+1 < 1$$

$$\Rightarrow -2 < 4x < 0$$

$$\Rightarrow -\frac{1}{2} < x < 0$$

∴ Given series is abs. cgt. for $x \in \left(-\frac{1}{2}, 0\right)$.

Ques: For what value of x, the power series $1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots + \left(\frac{-1}{2}\right)^n (x-2)^n + \cdots$ converges? What is its sum?

Soln: $u_n = \left(\frac{-1}{2}\right)^n (x-2)^n$

$$|u_n| = \frac{1}{2^n}\left|(x-2)^n\right|$$

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{2^n} \frac{|(x-2)^n|}{|(x-2)^{n+1}|} \cdot 2^{n+1}$$

$$= \lim_{n\to\infty} \frac{2}{|x-2|} = \frac{2}{|x-2|}$$

$\therefore$ By Ratio test, $\Sigma|u_n|$ is cgt.

if $\frac{2}{|x-2|} < 1 \Rightarrow |x-2| < 2$

$$\Rightarrow -2 < x-2 < 2$$

$$\Rightarrow 0 < x < 4$$

$\therefore$ $\Sigma|u_n|$ is cgt. for $0 < x < 4$

Since every abs. cgt. series is cgt.

$\rightarrow$ Σu_n is cgt. for $0 < x < 4$

Sum is an infinite G.P. with first term 1 and C.R. $\left|\frac{-1}{2}(x-2)\right| < \left|\frac{1}{2}(x-2)\right| < 1$

$$\text{Sum} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x} \quad \text{for } 0 < x < 4$$

Ans.