APPLICATION OF PD

(*) Taylor's & Maclaurin's Thm: for a fnx. of Two variables.

For one variable

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \cdots$$

For two variables

$$f(x+h, y+k) = f(x,y) + \left[h f_x \cancel{(a,b)} + k f_y \cancel{(a,b)}\right] + \frac{1}{2!}\left[h^2 f_{xx} \cancel{(a,b)} + k^2 f_{yy} \cancel{(a,b)} + 2hk f_{xy} \cancel{(a,b)}\right] + \frac{1}{3!}\left[h^3 f_{xxx} \cancel{(a,b)} + 3h^2 k f_{xxy} \cancel{(a,b)} + 3hk^2 f_{xyy} \cancel{(a,b)} + k^3 f_{yyy} \cancel{(a,b)}\right] + \cdots$$

$\Longrightarrow$ Cor 1. $\rightarrow$ Put $x = a$, $y = b$.

$$f(a+h, b+k) = f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hk f_{xy}(a,b)\right] + \cdots$$

$\Longrightarrow$ Cor 2. $\rightarrow$ Put $a+h = x$, $b+k = y$.

$\Rightarrow h = x - a$, $k = y - b$.

By Cor 1.

$$f(x,y) = f(a,b) + \left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right] + \cdots$$

venus

Date
Page

⟹ Cor 3: Putting $a=0$, $b=0$ in Cor 2.

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] + \ldots$$

This is called Maclaurin's Thm.

NOTE 1: → Cor 3 is used to expand $f(x,y)$ in powers of x & y (or to expand $f(x,y)$ in nbd of origin $(0,0)$).

NOTE 2: → Cor 2 is used to expand $f(x,y)$ in the nbd of a pt. (a,b).

Ques: Expand $e^x \sin y$ in powers of x & y as far as terms of third degree.

Soln:

$f(x,y) = e^x \sin y$	$f(0,0) = 0$
$f_x = e^x \sin y$	$f_x(0,0) = 0$
$f_y = e^x \cos y$	$f_y(0,0) = 1$
$f_{xx} = e^x \sin y$	$f_{xx}(0,0) = 0$
$f_{yy} = -e^x \sin y$	$f_{yy}(0,0) = 0$
$f_{xy} = e^x \cos y$	$f_{xy}(0,0) = 1$
$f_{xxx} = e^x \sin y$	$f_{xxx}(0,0) = 0$
$f_{xxy} = e^x \cos y$	$f_{xxy}(0,0) = 1$
$f_{xyy} = -e^x \sin y$	$f_{xyy}(0,0) = 0$
$f_{yyy} = -e^x \cos y$	$f_{yyy}(0,0) = -1$

$$\therefore f(x,y) = 0 + x\cdot 0 + y\cdot 1 + \frac{1}{2!}[x^2(0) + y^2(0) + 2xy(1)] + \frac{1}{3!}[x^3(0) + 3x^2y + 3xy^2(0) - y^3] + \ldots$$

$$f(x,y) = y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + \ldots$$ Ans.

Q. Expand e^{xy} at $(1,1)$.

Solu. $f(x,y) = e^{xy}$ | $f(1,1) = e$

$f_x = y e^{xy}$ | $f_x(1,1) = e$

$f_y = x e^{xy}$ | $f_y(1,1) = e$

$f_{xx} = y^2 e^{xy}$ | $f_{xx}(1,1) = e$

$f_{yy} = x^2 e^{xy}$ | $f_{yy}(1,1) = e$

$f_{xy} = e^{xy} + xy e^{xy}$ | $f_{xy}(1,1) = 2e$.

$$\therefore f(x,y) = f(1,1) + (x-1) f_x(1,1) + (y-1) f_y(1,1) + \frac{1}{2!}\Big[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1) f_{xy}(1,1)\Big] + \cdots$$

$$= e\Big[1 + x-1 + y-1 + \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2 + 2(x-1)(y-1) + \cdots\Big].$$

Ans.

Ques. Find the expansion of fnx. $e^x \log(1+y)$ in nbd. of pt. $(0,0)$ upto first six terms.

Solu. $f(x,y) = e^x \log(1+y)$ | $f(0,0) = 0$

$f_x = e^x \log(1+y)$ | $f_x(0,0) = 0$

$f_y = \dfrac{e^x}{1+y}$ | $f_y(0,0) = 1$

$f_{xx} = e^x \log(1+y)$ | $f_{xx}(0,0) = 0$

$f_{yy} = \dfrac{-e^x}{(1+y)^2}$ | $f_{yy}(0,0) = -1$

$f_{xy} = \dfrac{e^x}{1+y}$ | $f_{xy}(0,0) = 1$

$f_{xxx} = e^x \log(1+y)$ | $f_{xxx}(0,0) = 0$

$f_{xxy} = \dfrac{e^x}{1+y}$ | $f_{xxy}(0,0) = 1$

$$f_{xyy} = \frac{-e^x}{(1+y)^2} \qquad f_{xyy}(0,0) = -1$$

$$f_{yyy} = \frac{2e^x}{(1+y)^3} \qquad f_{yyy}(0,0) = 2$$

$$\therefore\ e^x \log(1+y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)\right] + \cdots$$

$$= 0 + 0 + y + \frac{1}{2!}\left[0 - y^2 + 2xy\right] + \frac{1}{3!}\left[0 + 3x^2y - 3xy^2 + 2y^3\right] + \cdots$$

$$e^x \log(1+y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} + \cdots$$

Ans.

Ques: Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in powers of h, k upto and inclusive of second degree terms.

Solu- $f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$

$$f(x,y) = \frac{xy}{x+y}$$

$$f_x = \frac{(x+y)y - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

$$f_y = \frac{x^2}{(x+y)^2}, \quad f_{xx} = -\frac{2y^2}{(x+y)^3},$$

$$f_{yy} = -\frac{2x^2}{(x+y)^3}, \quad f_{xy} = \frac{2xy}{(x+y)^3}$$

$$\therefore f(x+h, y+k) = f(x,y) + (h f_x + k f_y) + \frac{1}{2!}\left[h^2 f_{xx} + k^2 f_{yy} + 2hk f_{xy}\right] + \dots$$

$$= \frac{xy}{x+y} + \frac{hy^2 + x^2 k}{(x+y)^2} + \frac{1}{2!}\left[\frac{-2h^2 y^2 - 2k^2 x^2 + 4hk\,xy}{(x+y)^3}\right] + \dots$$

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \frac{hy^2 + kx^2}{(x+y)^2} - \frac{h^2y^2 + k^2x^2 - 2hkxy}{(x+y)^3} + \dots$$

Ans.

Ques Expand $f(x,y) = \sin xy$ in powers of $(x-1)$ and $(y - \frac{\pi}{2})$ upto second degree.

Soln

$f(x,y) = \sin xy$	$f(1, \pi/2) = 1$
$f_x = y \cos xy$	$f_x(1, \pi/2) = 0$
$f_y = x \cos xy$	$f_y(1, \pi/2) = 0$
$f_{xx} = -y^2 \sin xy$	$f_{xx}(1, \pi/2) = -\pi^2/4$
$f_{yy} = -x^2 \sin xy$	$f_{yy}(1, \pi/2) = -1$
$f_{xy} = -xy \sin xy + \cos xy$	$f_{xy}(1, \pi/2) = -\pi/2$.

$$\therefore \sin xy = 1 + (x-1)0 + (y - \pi/2)0 + \frac{1}{2!}\left[(x-1)^2\left(\frac{-\pi^2}{4}\right) + (y - \pi/2)^2(-1) + 2(x-1)(y-\pi/2)(-\pi/2)\right] + \dots$$

$$\sin xy = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{1}{2}(y - \pi/2)^2 - (x-1)(y-\pi/2)\frac{\pi}{2} + \dots$$

Ans.

Venus
Date
Page

Ques: Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's thm.

Soln:

$f(x,y) = x^2y + 3y - 2$	$f(1,-2) = -10$
$f_x = 2xy$	$f_x(1,-2) = -4$
$f_y = x^2 + 3$	$f_y(1,-2) = 4$
$f_{xx} = 2y$	$f_{xx}(1,-2) = -4$
$f_{yy} = 0$	$f_{yy}(1,-2) = 0$
$f_{xy} = 2x$	$f_{xy}(1,-2) = 2$
$f_{xxx} = 0$	$f_{xxx}(1,-2) = 0$
$f_{xxy} = 2$	$f_{xxy}(1,-2) = 2$
$f_{xyy} = 0$	$f_{xyy}(1,-2) = 0$
$f_{yyy} = 0$	$f_{yyy}(1,-2) = 0$.

$$\therefore \quad x^2y + 3y - 2 = -10 + (x-1)(-4) + (y+2)(4) + \frac{1}{2!}\left[(x-1)^2(-4) + (y+2)^2 0 + 2(x-1)(y+2)2\right] + \frac{1}{3!}\left[3(x-1)^2(y+2)2\right]$$

$$x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Ans.

Errors and Approximations

$$z = f(x, y)$$

$$z + \delta z = f(x + \delta x, y + \delta y).$$

$$\Rightarrow \delta z = f(x+\delta x, y+\delta y) - f(x, y)$$

$$\Rightarrow \delta z = \cancel{f(x,y)} + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \dots - \cancel{f(x,y)}$$

$$\Rightarrow \delta z = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} \quad \text{(Approx.}$$

Hence $dz = dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y}$.

$$\therefore \boxed{dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}.$$

NOTE 1: $\Longrightarrow$ If dx is error in x, then

$$\boxed{\begin{aligned} \text{relative error} &= \frac{dx}{x} \\ \text{\% age error} &= \frac{dx}{x} \times 100 \end{aligned}}$$

Ques: Find the %age error in the area of an ellipse when an error of +1 percent is made in measuring the major & minor axis.

Soln.

$$A = \pi ab$$

$$\log A = \log \pi + \log a + \log b$$

Differentiating, we get

$$\frac{dA}{A} = \frac{da}{a} + \frac{db}{b}$$

$$\frac{dA}{A} \times 100 = \frac{da}{a} \times 100 + \frac{db}{b} \times 100$$

$$= 1 + 1$$

Error in area $= 2\%$

Q. If $P = \frac{E^2}{R}$, then show that % error in P = 2% error in E − 1% error in R. Hence find approximate % change in P when E is increased by 3% and R is ↓ by 2%.

Soln.

$$P = \frac{E^2}{R}$$

$$\log P = 2\log E - \log R$$

$$\frac{dP}{P} = 2\frac{dE}{E} - \frac{dR}{R}$$

$$\frac{dP}{P} \times 100 = 2\frac{dE}{E} \times 100 - \frac{dR}{R} \times 100$$

⇒ % error in P = 2% error in E − 1% error in R.

Given $\frac{dE}{E} \times 100 = 3$

$$\frac{dR}{R} \times 100 = -2$$

$$\therefore \frac{dP}{P} \times 100 = 2(3) - (-2)$$
$$= 8$$

$$dP = \frac{8}{100} P$$

⇒ Change in P = 8%.

Ans.

Ques. The period T of a simple pendulum of length l is given by $T = 2\pi\sqrt{l/g}$. Find

(i) error

(ii) % error (relative error)

made in computing T by using $l = 2ft$, $g = 32 ft/sec^2$ if the true values are $l = 1.95$ and $g = 32.2 ft/sec^2$.

Date
Page

Soln. $T = 2\pi\sqrt{\frac{l}{g}}$.

$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$.

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l} - \frac{1}{2}\frac{dg}{g} \quad \text{---} \; (1)$$

Given $l = 1.95$, $l + dl = 2 \Rightarrow dl = 0.05$

$g = 32.2$, $g + dg = 32 \Rightarrow dg = -0.2$

(i) $\frac{dT}{T} = \frac{1}{2}\left[\frac{0.05}{1.95} + \frac{0.2}{32.2}\right]$

$= \frac{1}{2}(0.025 + 0.0062)$

$dT = 0.0156\, T$

$= 0.0156 \times 2 \times \frac{22}{7}\sqrt{\frac{1.95}{32.2}}$

$\boxed{dT = 0.024}$

(ii) % error $\frac{dT}{T} \times 100 = 1.56$

Relative error $\frac{dT}{T} = 0.0156$.

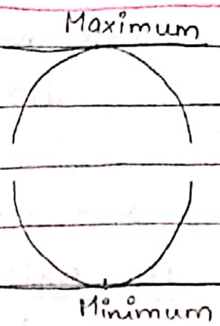
Ans

⟹ Important terms

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$,

$s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$.

Maxima & Minima of Fns. of two variable.

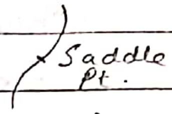


A fnx. $f(x,y)$ is said to have a max. value at $x=a$, $y=b$ if $f(a,b) > f(a+h, b+k)$, for small in independent values of h and k (+ve or -ve).

A fnx. $f(x,y)$ is s.t. have a min. value at $x=a$, $y=b$ if $f(a,b) < f(a+h, b+k)$ for small & ind. values of h & k (+ve or -ve).

$\Longrightarrow$ $f(x,y)$ has max. or min. value when $\boxed{\Delta f > 0 \text{ or } < 0}$.

Saddle Point



A pt. (a,b) is s.t.b. saddle pt. of a surface if tangent plane at (a,b) is horizontal and surface descends in some direction and ascends in the other direction.

Extreme Value

A max. or a min. value of a fnx. is called its extreme value.

Working Rule to find extreme value of $z=f(x,y)$

① Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

② Solve $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$. Let $(a,b), (c,d) \ldots$

③ For each soln. find r, s, t.

④ (a) $rt - s^2 > 0$ & $r < 0$ $\Rightarrow$ z has max. value.
(b) $rt - s^2 > 0$ & $r > 0$ $\Rightarrow$ z has min. value.
(c) $rt - s^2 < 0$ $\Rightarrow$ No extreme value
(d) $rt - s^2 = 0$ $\Rightarrow$ Requires further investigation.

Date
Page

Ques: Examine the fnx. x^3+y^3-3axy for maxima and minima.

Sol
Soln. $f(x,y) = x^3+y^3-3axy$

$f_x = 3x^2-3ay$, $f_y = 3y^2-3ax$

$r = 6x$, $t = 6y$, $s = -3a$

For extreme values

$f_x = 0$ & $f_y = 0$

$x^2 - ay = 0$ —— ① $y^2 - ax = 0$ —— ②

① $\Rightarrow y = \frac{x^2}{a}$

② $\Rightarrow \frac{x^4}{a^2} = ax \Rightarrow \boxed{x = 0, a}$

When $x=0$, $y=0$ & $x=a$, $y=a$

$\therefore$ Extreme values are $(0,0)$, (a,a).

At $(0,0)$ $rt - s^2 = 36xy - 3a^2$
$= -3a^2 < 0$

$\Rightarrow$ No extreme value at $(0,0)$.

At (a,a) $rt - s^2 > 0$, $r = 6a$

If $a > 0$, $r > 0$

$\Rightarrow z$ has min. value at (a,a).

If $a < 0$, $r < 0$

$\Rightarrow z$ has max. value at (a,a).

Min & Max value $= a^3 + a^3 - 3a^3$
$= -a^3$

Ans.

Q. Locate stationary pts. of $x^4+y^4-2x^2+4xy-2y^2$ + determine their nature.

Soln. $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

$f_x = 4x^3-4x+4y$

$f_y = 4y^3-4y+4x$

$r = 12x^2-4$

$t = 12y^2-4 \qquad s = 4$

$rt-s^2 = (12x^2-4)(12y^2-4)-16$

- For extreme values

$f_x=0 \Rightarrow x^3-x+y=0$ —— ①

$f_y=0 \Rightarrow y^3+x-y=0$ —— ②

① + ② $\Rightarrow x^3=-y^3$

$\Rightarrow \boxed{y=-x}$

① $\Rightarrow x^3-x-x=0 \rightarrow x^3=2x$

$\Rightarrow x=0, \pm\sqrt{2}$

$\therefore$ Extreme values are $(0,0), (\sqrt{2},-\sqrt{2}), (-\sqrt{2},\sqrt{2})$.

At (0,0) $\qquad rt-s^2=0$

Further investigation is needed.

~~At $(\sqrt{2},-\sqrt{2})$~~ $\qquad$ Now $f(0,0)=0$

~~$rt-s^2 = 20$~~

$f(x,y) = x^4+y^4-2(x-y)^2$

When h + k are small

$f(h,k) = 2h^4 > 0$ for $h=k$

+ $f(h,k) = -2(h-k)^2 < 0$ for $h \neq k$

(neglecting $h^4+k^4 <<<< 0$)

$\Rightarrow f(0,0) < f(h,k)$ for $h=k$

$f(0,0) > f(h,k)$ for $h \neq k$

$\Rightarrow$ No extreme value at (0,0).

$4+4-4-8-4$ $400-16$

At $(\sqrt{2}, -\sqrt{2})$ $rt - s^2 > 0$

So $r > 0$

$\therefore f(x,y)$ has min value at $(\sqrt{2}, -\sqrt{2})$

Min. value $= 4+4-4-8-4$

$= -8$

At $(-\sqrt{2}, \sqrt{2})$ $rt - s^2 > 0$

$r > 0$

$\therefore f(x,y)$ has min. value at $(-\sqrt{2}, \sqrt{2})$.

Q. The temp. T at any pt. (x,y,z) in space $T = 400xyz^2$. Find highest temp. on the surface of unit spere $x^2+y^2+z^2=1$.

Soln: $z^2 = 1 - x^2 + y^2$

$\therefore$ $T = 400xy(1-x^2-y^2)$.

For extreme value

$$T_x = 400\left[y(1-x^2-y^2) + xy(-2x)\right]$$

$$T_x = 400y(1-3x^2-y^2)$$

$$T_y = 400x(1-x^2-3y^2).$$

$$r = 400y(-6x) = -2400xy$$

$$s = 400(1-3x^2-y^2) + 400y(-2y)$$

$$s = 400(1-3x^2-3y^2)$$

$$t = -2400xy.$$

$$rt - s^2 = (2400xy)^2 - 400^2(1-3x^2-3y^2)^2.$$

$T_x = 0 \Rightarrow y(1-3x^2-y^2) = 0$ —— ①

$T_y = 0 \Rightarrow x(1-x^2-3y^2) = 0$ —— ②

① $\Rightarrow$ $y = 0$ or $1-3x^2-y^2 = 0$

② $\Rightarrow$ $x = 0$ or $1-x^2-3y^2 = 0$

$\therefore$ Stationary pts. are $(0,0)$, $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$,

$(\pm 1, 0)$, $(0, \pm 1)$.

At $(0,0)$ $\quad rt-s^2 < 0$

$\Rightarrow$ No extreme point.

At $(\pm 1, 0)$ $\quad r = 0$

$rt - s^2 < 0$

$\Rightarrow$ No extreme point.

At $(0, \pm 1)$ $\quad r = 0$

$rt - s^2 < 0$

$\Rightarrow$ No extreme point.

At $\left(\frac{1}{2}, \frac{1}{2}\right)$ $\quad r < 0$

$rt - s^2 > 0$

$\Rightarrow$ T has max. at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

At $\left(\frac{-1}{2}, \frac{-1}{2}\right)$ $\quad r < 0$

$rt - s^2 > 0$

$\Rightarrow$ T is max. at $\left(\frac{-1}{2}, \frac{1}{2}\right)$.

At $\left(\frac{1}{2}, \frac{-1}{2}\right)$ $\quad r > 0$

$rt - s^2 > 0$

$\Rightarrow$ T is min. at $\left(\frac{1}{2}, \frac{-1}{2}\right)$.

At $\left(\frac{-1}{2}, \frac{1}{2}\right)$ $\quad r > 0$

$rt - s^2 > 0$

$\Rightarrow$ T is min. at $\left(\frac{-1}{2}, \frac{1}{2}\right)$.

$\therefore$ T has min. value at $\left(\frac{1}{2}, \frac{-1}{2}\right)$ & $\left(\frac{-1}{2}, \frac{1}{2}\right)$.

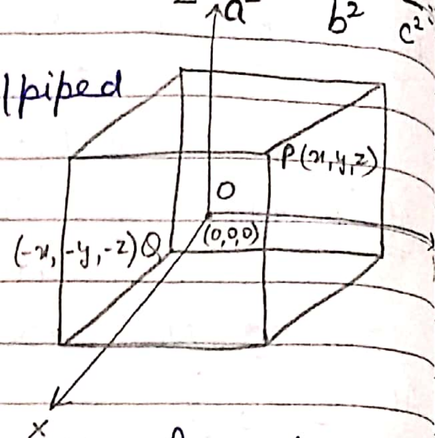
T has max. value at $\left(\frac{1}{2}, \frac{1}{2}\right)$ & $\left(\frac{-1}{2}, \frac{-1}{2}\right)$.

$\therefore \quad T_{max} = 400\left(\frac{1}{4}\right)\left(1 - \frac{1}{4} - \frac{1}{4}\right)$

$\boxed{T_{max} = 50}$

Ques. Find the vol. of largest rectangular ||piped that can be inscribed in ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$

Soln. Let (x, y, z) be a vertex of ||piped then it lies on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)



As rectangular ||piped inscribed in the ellipsoid is the largest
$\therefore$ $(0,0,0)$ is also centre of ||piped.
If (x,y,z) are cordinates of one of the vertices of ||piped. Then coordinates of its opposite vertex is $(-x,-y,-z)$.
$\therefore$ Sides of ||piped are of dimensions $2x, 2y, 2z$.
$\therefore$ Volume $V = 2x \cdot 2y \cdot 2z$

$$V = 8xyz$$

$$V^2 = 64x^2y^2z^2$$

$$= 64x^2y^2c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$\Rightarrow f(x,y) = 64c^2\left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right)$$

$$f_x = 64c^2\left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right)$$

$$f_y = 64c^2\left(2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right).$$

$$r = 64c^2\left(2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right)$$

$$s = 64c^2\left(2x^2 - \frac{8x^4}{a^2} - \frac{12x^2y^2}{b^2}\right)$$

$$t = 64c^2\left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right),$$

Now $f_x = 0$ and $f_y = 0$

$$\Rightarrow 128c^2xy^2\left(1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}\right)=0$$

and $$128c^2x^2y\left(1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}\right)=0$$

$$\Rightarrow 1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}=0 \quad \text{—— (1)}$$

$$1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}=0 \quad \text{—— (2)}$$

(1) − (2) $\quad y = \frac{bx}{a}$

$\therefore$ (1) $\Rightarrow \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}$.

and $z^2 = c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b}\right) = \frac{c^2}{3} \Rightarrow z = \frac{c}{\sqrt{3}}$.

$\therefore$ Stationary pts. is $x = \frac{a}{\sqrt{3}}, \; y = \frac{b}{\sqrt{3}}$.

At $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$, $\quad r = -\frac{512}{9}b^2c^2 < 0$

$t = -\frac{512}{9}a^2c^2, \quad s = -\frac{256}{9}abc^2$.

$$rt - s^2 = \left(\frac{512}{9}\right)^2 a^2b^2c^4 - \left(\frac{256}{9}\right)^2 a^2b^2c^4 \; . > 0$$

$\therefore$ V^2 & hence V is max. at

$$x = \frac{a}{\sqrt{3}}, \; y = \frac{b}{\sqrt{3}}, \; z = \frac{c}{\sqrt{3}}.$$

$\therefore$ Maximum volume $= 8\,\frac{a}{\sqrt{3}}\,\frac{b}{\sqrt{3}}\,\frac{c}{\sqrt{3}}$

$$= \frac{8abc}{3\sqrt{3}}$$

Ans

Date
Page

Q. The sum of three +ve nos. is const. Prove that their product is max. when they are equal.

Soln. Let three +ve nos. be x, y, z

Given $x+y+z = k$

Product $P = xyz \Rightarrow P = xy(k-x-y)$.

$P_x = y(k-x-y) - xy$

$\Rightarrow P_x = y(k-2x-y)$

$P_y = x(k-x-2y)$

$r = P_{xx} = -2y$, $s = k-2x-2y$.

$t = -2x$

For extreme values $P_x = 0$ & $P_y = 0$

$\Rightarrow y(k-2x-y) = 0$	$x(k-x-2y) = 0$
$\Rightarrow y=0$ or $k-2x-y=0$	$\Rightarrow x=0$ or $k-x-2y=0$
$\Rightarrow x = \frac{k}{3}$,	$y = \frac{k}{3}$

$\therefore$ Stationary pts. are $(0,0)$, $\left(\frac{k}{3}, \frac{k}{3}\right)$.

At $(0,0)$ $rt - s^2 = 4xy - (k-2x-2y)^2$

< 0

$\therefore$ P has no extreme value at $(0,0)$.

At $(k/3, k/3)$

$rt - s^2 = \frac{4k^2}{9} - \left(k - \frac{2k}{3} - \frac{2k}{3}\right)^2$

$= \frac{k^2}{3} > 0$

Also $r < 0$

$\therefore$ P has max. value at $\left(\frac{k}{3}, \frac{k}{3}\right)$

$P_{max} = \frac{k}{3}\frac{k}{3}\frac{k}{3} = \frac{k^3}{27}$ at $x=y=z=\frac{k}{3}$

Scanned by CamScanner

Lagrange's Method of Undetermined Multipliers

To find the max. or min. values of a fnx. of three (or more) variables, when the variables are not independent but are connected by some given relation, we try to convert given fnx. to one, having least no. of ind. variables with the help of given relation.

When this is not practicable (i.e. variables can't be converted), we use Lagrange's method.

Let $f(x,y,z)$ be a fnx. of x,y,z which is to be examined for maximum or minimum value. Let variables x,y,z be connected by the relation $\phi(x,y,z) = 0$ —— (1)

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

This is Auxillary eqn.

For stationary values of $F(x,y,z)$

$$dF = 0$$

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0. \quad \text{—— (2)}$$

Advantages & Disadvantages

Advantages → The stationary values of f can be determined from ϕ & (2), even without determining x,y,z explicitly.

→ This method can be extended to a fnx. of several variables & subject to any no. of constraints.

§ *Disadvantages*

This method helps us to find stationary pts. but does not tell us whether that pt. is max. or min.

For this we find d^2f.

$$d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx.$$

If $d^2f > 0$, min. value

$d^2f < 0$, max. value.

Ques Find the max. & min. distances of the pt. $(3, 4, 12)$ from sphere $x^2+y^2+z^2 = 1$.

Soln. Distance $= \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$. —— (1)

Let $f(x,y,z) = (x-3)^2 + (y-4)^2 + (z-12)^2$.

$\phi(x,y,z) = x^2+y^2+z^2-1 = 0$. —— (2)

Consider Lagrange's fnc.

$F = f + \lambda\phi$

$F = (x+3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-1)$

For stationary values $dF = 0$

$$\Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

$$\Rightarrow [2(x-3) + 2\lambda x]\,dx + [2(y-4) + 2\lambda y]\,dy + [2(z-12) + 2\lambda z]\,dz = 0$$

$\Rightarrow$ $2(x-3) + 2\lambda x = 0$ —— (3)

$2(y-4) + 2\lambda y = 0$ —— (4)

$2(z-12) + 2\lambda z = 0$ —— (5)

③, ④ & ⑤ give values of x, y, z resp.

$x = \frac{3}{1+\lambda}$, $y = \frac{4}{1+\lambda}$, $z = \frac{12}{1+\lambda}$

③ x + ④ y + ⑤ z

$\Rightarrow 2(x^2+y^2+z^2) - 6x - 8y - 24z + 2\lambda(x^2+y^2+z^2) = 0$

$\Rightarrow 2 - 6x - 8y - 24z + 2\lambda = 0$ (Using ②)

$\Rightarrow 6x + 8y + 24z = 2 + 2\lambda$

$\Rightarrow 3x + 4y + 12z = 1 + \lambda$ —— ⑥

Put values of x, y, z in ⑥

$$\frac{9}{1+\lambda} + \frac{16}{1+\lambda} + \frac{144}{1+\lambda} = 1+\lambda$$

$$(1+\lambda)^2 = 169$$

$$\Rightarrow 1+\lambda = \pm 13$$

$$\Rightarrow \boxed{\lambda = 12, -14}$$

$\therefore$ $x = \frac{3}{13}$, $y = \frac{4}{13}$, $z = \frac{12}{13}$ (for $\lambda = 12$)

$x = \frac{-3}{13}$, $y = -\frac{4}{13}$, $z = -\frac{12}{13}$ (for $\lambda = -14$)

$\therefore$ Two pts. are $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ & $\left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$

$\therefore$ Max. dist $= \sqrt{\left(\frac{36}{13}\right)^2 + \left(\frac{48}{13}\right)^2 + \left(\frac{144}{13}\right)^2}$

$= \frac{156}{13} = 12$

Min. dist. $= \sqrt{\left(\frac{42}{13}\right)^2 + \left(\frac{56}{13}\right)^2 + \left(\frac{168}{13}\right)^2}$

$= \frac{182}{13} = 14$

Venus Date Page

Q. Find the min. value of $x^2+y^2+z^2$ given that $xyz = a^3$.

Soln. Let $u = x^2+y^2+z^2$

$\phi = xyz - a^3$ —— (1)

Consider Lagrange's fnx.

$F = u + \lambda\phi$

$F = x^2+y^2+z^2 + \lambda(xyz - a^3)$ —— (2)

For stationary pts, $dF = 0$

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

$\Rightarrow (2x + yz\lambda)dx + (2y + xz\lambda)dy + (2z + xy\lambda)dz = 0$

$\Rightarrow 2x + \lambda yz = 0$ —— (3)

$2y + \lambda zx = 0$ —— (4)

$2z + \lambda xy = 0$ —— (5)

$x(3) + y(4) + z(5)$

$\Rightarrow 2(x^2+y^2+z^2) + 3\lambda xyz = 0$

$2u + 3\lambda a^3 = 0$

$\Rightarrow \lambda = -\frac{2u}{3a^3}$ —— (6)

From (3), (4) & (5), we get

$x = \frac{u}{3a^3}yz, \quad y = \frac{u}{3a^3}zx, \quad z = \frac{u}{3a^3}xy$

From (1)

$$\frac{u}{3a^3}yz \cdot \frac{u}{3a^3}zx \cdot \frac{u}{3a^3}xy = a^3$$

$$\Rightarrow \frac{u^3(xyz)^2}{27a^9} = a^3$$

$$\Rightarrow \frac{u^3a^6}{27a^9} = a^3 \Rightarrow \boxed{u = 3a^2}$$

This is the extreme value of u.

For max. or min. value, we find d^2f.

$$d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx.$$

$$d^2f = 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 2\left[z\lambda\,dx\,dy + x\lambda\,dy\,dz + y\lambda\,dz\,dx\right]$$

Given $\phi(x,y,z) = xyz - a^3 = 0$

$$d^2\phi = 0$$

$$\therefore \phi_{xx}(dx)^2 + \phi_{yy}(dy)^2 + \phi_{zz}(dz)^2 + 2\phi_{xy}\,dx\,dy + 2\phi_{yz}\,dy\,dz + 2\phi_{zx}\,dz\,dx = 0$$

$$\Rightarrow 0+0+0+2\left[z\,dx\,dy + x\,dy\,dz + y\,dz\,dx\right] = 0$$

$$\therefore d^2f = 2\left[(dx)^2 + (dy)^2 + (dz)^2\right] > 0$$

$\therefore$ u is minimum & min. value of $u = 3a^2$.

Ques: Use Lagrange's method to find the min. value of $x^2+y^2+z^2$ subject to the conditions $x+y+z=1$ & $xyz+1=0$.

Soln: Consider Lagrange's fnx.

$$f = x^2+y^2+z^2 + \lambda(x+y+z-1) + \mu(xyz+1) \quad \text{--- (1)}$$

$$df = 0 \Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

$$\Rightarrow (2x+\lambda+\mu yz)dx + (2y+\lambda+\mu xz)dy + (2z+\lambda+\mu xy)dz = 0$$

$$\Rightarrow 2x+\lambda+\mu yz = 0 \quad \text{--- (2)}$$

$$2y+\lambda+\mu xz = 0 \quad \text{--- (3)}$$

$$2z+\lambda+\mu xy = 0 \quad \text{--- (4)}$$

venu

Date
Page

$(2) - (3) \Rightarrow 2(x-y) + \mu z(y-x) = 0$

$\Rightarrow (x-y)(2-\mu z) = 0$

$\Rightarrow$ Either $x-y=0$ or $2-\mu z = 0$

$\Rightarrow x = y$ or $z = \frac{2}{\mu}$.

Ilky from (3) & (4) either $y = z$ or $x = \frac{2}{\mu}$

& from (2) & (4) either $x = z$ or $y = 2/\mu$

Combining we get

$x = y = z$ or $\mu = \frac{2}{x} = \frac{2}{y} = \frac{2}{z}$.

Given $x + y + z = 1$

$\Rightarrow x = 1/3, \; y = 1/3, \; z = \frac{1}{3}$

and $\mu = \frac{2}{1/3} = 6$

$\therefore$ Stationary pt. is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Also $d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx$.

$= 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 2\mu z\,dx\,dy + 2\mu x\,dy\,dz + 2\mu y\,dz\,dx$

$= 2\left[dx^2 + dy^2 + dz^2 + 2\,dx\,dy + 2\,dy\,dz + 2\,dz\,dx\right]$

$= 2\left[dx + dy + dz\right)^2$

$d^2f > 0$

$\therefore f$ is min. at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ & min. value

of $x^2 + y^2 + z^2$ is $\frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$

Ans.

Geometrical Meaning of Partial Derivatives

$z = f(x,y)$ represents a surface S.
If $y = k$, constt., then it represents a plane $\parallel$ to zx-plane.

$\therefore$ $z = f(x,y)$ & $y = k$ represent a plane curve C which is the section of S by $y = k$.

$\frac{\partial z}{\partial x}$ represents slope of tangent to C at (x, k, z).

$\therefore$ $\frac{\partial z}{\partial x}$ represents slope of tangent drawn to the curve of intersection of the surface $z = f(x,y)$ & a plane $\parallel$ to zx-plane.

|||ly $\frac{\partial z}{\partial y}$ represents slope of tangent drawn to the curve of intersection of the surface $z = f(x,y)$ & a plane $\parallel$ to the yz-plane.

Tangent & Normal to a surface.

Eqn. of tangent plane to surface $f(x,y,z) = 0$.

$$\frac{\partial f}{\partial x}(X - x) + \frac{\partial f}{\partial y}(Y - y) + \frac{\partial f}{\partial z}(Z - z) = 0 .$$

where (X, Y, Z) are current coordinates of any pt. on this tangent plane.

Eqn. of normal plane

$$\frac{X - x}{f_x} = \frac{Y - y}{f_y} = \frac{Z - z}{f_z}.$$

Venus Date Page

Q. Find the eqn. of tangent plane & normal line to the surface $xyz = a^3$ at (x_1, y_1, z_1).

Soln. $f(x,y,z) = xyz - a^3 = 0$.

$f_x = yz$, $f_y = xz$, $f_z = xy$.

At (x_1, y_1, z_1), $f_x = y_1 z_1$, $f_y = x_1 z_1$, $f_z = x_1 y_1$

Eqn. of tangent plane

$(x - x_1) y_1 z_1 + (y - y_1) x_1 z_1 + (z - z_1) x_1 y_1 = 0$

$$\Rightarrow \frac{x - x_1}{x_1} + \frac{y - y_1}{y_1} + \frac{z - z_1}{z_1} = 0$$

$$\Rightarrow \boxed{\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 3}$$

Eqn. of normal plane

$$\frac{x - x_1}{y_1 z_1} = \frac{y - y_1}{x_1 z_1} = \frac{z - z_1}{x_1 y_1}.$$

Q. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$ if $\frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$.

Soln. Let plane $ax + by + cz + d = 0$ — ① touches the surface

$F(x,y,z) = px^2 + qy^2 + 2z = 0$ at (x_1, y_1, z_1) — ②

② $\Rightarrow$ $F_x = 2px$, $F_y = 2qy$, $F_z = 2$

At (x_1, y_1, z_1) $\Rightarrow$ $F_x = 2px_1$, $F_y = 2qy_1$, $F_z = 2$

Equ. of tangent plane

$$(x-x_1)\,2px_1 + (y-y_1)\,2qy_1 + (z-z_1)\,2 = 0$$

$$\Rightarrow pxx_1 + qyy_1 + z = px_1^2 + qy_1^2 + z_1$$

As (x_1, y_1, z_1) lies on ②

$$\therefore \quad px_1^2 + qy_1^2 + 2z_1 = 0$$

$$\Rightarrow px_1^2 + qy_1^2 + z_1 = -z_1$$

$\therefore$ Equ. of tangent plane is

$$pxx_1 + qyy_1 + z = -z_1$$

$$\Rightarrow pxx_1 + qyy_1 + z + z_1 = 0$$

which is same as ①

Comparing the coeffs.

$$\frac{px_1}{a} = \frac{qy_1}{b} = \frac{1}{c} = \frac{z_1}{d}$$

$$\therefore x_1 = \frac{a}{cp}, \quad y_1 = \frac{b}{cq}, \quad z_1 = \frac{d}{c}$$

$\therefore$ ② $\Rightarrow$

$$p\frac{a^2}{c^2p^2} + q\frac{b^2}{c^2q^2} + \frac{2d}{c} = 0$$

$$\Rightarrow \frac{a^2}{p} + \frac{b^2}{q} + 2dc = 0$$

Hence proved.